DEFENSE DOCUMENTATION CENTER

FOR

SCIENTIFIC AND TECHNICAL INFORMATION

CAMERON STATION. ALEXANDRIA. VIRGINIA

408 493

63-4-2

Columbia University
in the City of New York

DEPARTMENT OF CIVIL ENGINEERING
AND ENGINEERING MECHANICS

SEMINAR ON CONTINUUM MECHANICS
ARRANGED AS
A COURSE OF TEN LECTURES ON THE
PRINCIPLES OF CONTINUUM MECHANICS AS SEEN
IN THE LIGHT OF A GENERAL THEORY OF TRANSFORMATIONS

by

Karl Weissenberg

March 1963

Columbia University
in the City of New York

DEPARTMENT OF CIVIL ENGINEERING
AND ENGINEERING MECHANICS

SEMINAR ON CONTINUUM MECHANICS
ARRANGED AS
A COURSE OF TEN LECTURES ON THE
PRINCIPLES OF CONTINUUM MECHANICS AS SEEN
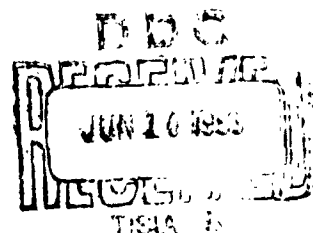IN THE LIGHT OF A GENERAL THEORY OF TRANSFORMATIONS

by

Karl Weissenberg

Preface

The Department of Civil Engineering and Engineering Mechanics of Columbia University, New York 27, New York, under the sponsorship of the Structural Mechanics Branch of the Office of Naval Research arranged a Seminar of ten lectures with the title "Principles of Continuum Mechanics in the Light of a General Theory of Transformations." These lectures are meant to serve as an introduction to the recent advances made when Continuum Mechanics was extended from the classical treatment of infinitesimally small deformations to a treatment of deformations of unlimited magnitude; they were presented in the Spring Semester 1962.

The author of this Report wishes to acknowledge with thanks the financial support received from the Office of Naval Research and the help and encouragement which Professor A. M. Freudenthal and Members of his Staff have given him throughout the whole series of lectures.

The Summary given below records the main results, together with the underlying ideas. A list of references is added of the papers published on the subject by the author of the Report and by his collaborators.

In continuum mechanics (as well as in every other section
of physics) the transformation theory starts from the recogni-
tion that every observation is essentially relative, in that it
specifies some relation between the observed and the observer,
and hence varies for the same observed object with a change
from one observer to another. In order to find the character-
istics of the observed object one has to eliminate, as far as
possible, the arbitrariness of the particular viewpoint of the
observer, and the transformation theory shows that this can be
achieved by combining the observations in various ways such
that the resultant combinations are invariant against changes
within larger and ever larger groups of observers. For
quantitative observations the observer will be represented by
a system of reference (i.e., a set of measuring instruments
comprising a system of space coordinates, yardsticks, time clocks,
balances, etc., with calibrated scales) and elimination of the
arbitrariness in the particular choice of the system of refer-
ence, and of the scale functions, will be achieved by combining
the observed quantitative data in various ways such that the
resultant combinations are invariant against transformations
within wider and ever wider groups of systems of reference.
These invariant combinations which are spotlighted by the trans-
formation theory are of primary importance in every section of

physics, including continuum mechanics for which it is shown
(Lectures I and II) that they are the basis for a formulation
of all the various theoretical concepts, laws and principles,
as well as for the design of the instrumentation used in ex-
periments. They also permit to theoretically predict and con-
trol certain apparently paradoxical phenomena of mechanical
behaviour, as illustrated in experiments. For the formulation
of invariant combinations, or invariants for short, the trans-
formation theory indicates a procedure which rests entirely on
the goniometry, i.e., the measurement of the angular distribu-
tions of the divers variables which coexist at any point in the
medium. The tensor calculus with its invariant, and co- and
contravariant representations of the angular distributions of
variables round a point by scalars, vectors, tensors and pseudo-
tensors (volume capacities and densities of tensors) as pre-
eminently suited to carry out such investigations, and is
accordingly used throughout this series of lectures. Classical
continuum mechanics has already used the tensor calculus and
has fully appreciated the importance of invariants of certain
kinds which remain unchanged under certain groups of transfor-
mations, including the group of symmetry operations of the
Euclidean space, the group of Galilei transformations of a
space-time continuum, and the group of similitude transformations

of the units of measurements of space, time and mass. In connection with the stresses and strains involved in continuum mechanics the modern development has added a new and more comprehensive kind of invariants which remain unchanged under the transformations of all the groups mentioned above, as well as under all transformations of the scale functions by which the stresses and strains happened to be measured. It is this comprehensive kind of invariants which, in conjunction with the general theory of transformations, is at the basis of the progress made in continuum mechanics in all its three branches concerned respectively with the Kinematics, Dynamics and Mechanical Properties of Continuous Media.

In the Kinematics (Lecture III) the general transformation theory is applied to a study of the invariants of the principle of preservation of continuity as revealed in the goniometry of continuous movement. The results of this study in combination with Lagrange's theorem of forces of restraint lead to a new model technique based on a mechanical device referred to as a "trellis" because in the two dimensional case it operates like the wellknown collapsible garden trellis. For the design and construction of the trellis model one considers a continuous deformation from an initial to a terminal position, then represents the change in position by a coordinate trans-

formation, and finally determines the invariants under this transformation. These invariants form a pattern of mutually intersecting lines of zero elongation which in the course of deformations can change their directions but must have in both the initial and terminal positions the same length and form, and the same points of intersection. The design of the trellis model then corresponds to the pattern of intersecting lines of zero elongation, and the construction provides for a framework in which the points of intersections are replaced by pinpointed universal joints and linked to one another by rigid rods replacing the lines of zero elongation between two neighboring junctions. The rigidity of the rods, i.e., their inextensibility and inflexibility, and the fixing of the pinpointed joints represent the infinitely strong Lagrange forces of restraint which offer no resistance against a continuous deformation from the initial to the terminal position but an infinitely strong resistance against any changes of the invariants which are postulated for the said deformation by the principle of preservation of continuity. The trellis model has been put to many practical uses, e.g., the solution of boundary problems, the precalculation of fatigue cracks, etc. In a further study of continuous movements (Lecture IV) laws are derived which allow the calculation of the superposition of such movements,

and of their resolutions into purely rigid and purely deforma-

tory components.

In the Dynamics consideration is given to the different

conceptions of forces (Newton, Lagrange, Einstein), and the dif-

ferent types (mass and traction forces), all of which can be

distinguished by their different transformation properties. A

study of the invariants of the principle of equilibrium of

forces, as revealed in the Goniometry of the forces, then leads

to the formulation of various equilibria all of which are found

to be valid for all continuous media and conditions, nd in-

variant against the group of Galilei transformations. The

said equilibria are used in the design of devices for the meas-

urement of forces,and for the solution of certain boundary

problems.

In the Mechanical Properties of Continuous Media it was

found necessary to study the invariants of the governing prin-

ciples separately in their various aspects. The experimental

aspect was concerned with the instrumentation (Lecture VI),

and here principles were formulated for the design of "ideal"

instruments capable of supplying all the information required

for a goniometric measurement of the mechanical properties.

Two instruments of a new type were discussed which approximated

the ideal. The subsequently discussed theoretical aspect dealt

with various new and exact definitions of the mechanical proper-
ties and pseudo-properties (Introduction to Lecture VII), and
with the establishment of a principle referred to as an Equa-
tion of State, that would apply to all materials and conditions,
and would allow a specification of all the said mechanical prop-
erties and pseudo-properties. From the thermodynamical point
of view (Lecture VII Continued) a Scalar Equation of State is
formulated as a law regulating the transformation of energy
which occurs when a continuous medium is subjected to mechanical
actions. From the said law a complete cycle of mutually inter-
connected theories is deduced which specify the various proper-
ties (and pseudoproperties) such as elasticity, viscosity, re-
laxation and retardation, etc. A different point of view is
then discussed in considering the Equation of State as a corre-
lation of the stresses and strains which coexist in every differ-
ential cell of the medium (Lecture VIII). For such a correlation
to be physically significant it must be invariant not only
against transformations of the coordinates of space and time,
but also against all transformations of the scale functions by
which the stresses and strains happen to be measured. An ex-
ample of such an invariant correlation is an appropriately de-
fined "anisotropic similitude" which is used for the development
of yet another new model technique, in generalization of the

classical one that had been based only on invariance against
transformations of the units of space, time and mass. Finally,
an Equation of State in Tensor form is proposed, and deduced
first by way of an analytical procedure (Lecture IX) and then
by way of a generalization of the classical linear laws (Lec-
ture X). Both ways lead to the same result with a stress-
strain relation which again is invariant not only against trans-
formations of the space and time coordinates, but also against
the transformations of the scale functions for the measurement
of the stress and strain. The anisotropic similitude is con-
tained as a special case in the said relation. The series of
lectures ends with a discussion of the usefulness of the Ten-
sorial Equation of State.

TABLE OF CONTENTS

Seminar on Continuum Mechanics

Lecture I

Survey of Continuum Mechanics

As an introduction to the course of lectures given
in the Seminar on Continuum Mechanics the whole field was
reviewed in the light of a general theory of transformations.
In this review the fundamental principles and experimental
techniques were discussed in three sections, dealing respec-
tively with the Kinematics, Dynamics, and Mechanical Proper-
ties of Continuous Media. In each of these sections the
application of the general Theory of Transformation has made
it possible to predict various phenomena of flow which had
previously been unexpected and often presented even a para-
doxical appearance. The phenomena were demonstrated in a
series of experiments supported by the projection of slides.

Seminar on Continuum Mechanics

Lecture II

The General Theory of Transformations

A. Introduction

1. The General Theory of Transformations as a Research
Tool in Continuum Mechanics

B. The Dependence of Quantitative Observations on the Observed
Object and on the Metrical System of Reference Acting as
Observer

1. The Arbitrariness of the Scale Functions of the Metrical
System of Reference, and the Elimination of the Arbi-
trariness by Way of Invariants and Co- and Contra-
variants

2. The Tensor Calculus as Key to the Goniometry

3. The Mechanical Symmetry

4. The Mechanical Similitude

C. The Principles of Continuum Mechanics in the Light of a
General Theory of Transformations

1. Approximation through the Use of Newton-Galilei Systems
of Reference

2. Principles and Model Techniques in a Comprehensive
Scheme of Continuum Mechanics

Seminar on Continuum Mechanics

Lecture II

The General Theory of Transformations

A. Introduction

1. The General Theory of Transformations as a Research
   Tool in Continuum Mechanics

The development of Physics as an exact science brought

about the need for a concise formalism by which one could inter-

connect all observations of physical phenomena with one another,

and with a mathematical scheme of equations from which one could

precalculate theoretically the experimental data resulting from

the said observations. Such a formalism was found in the gen-

eral theory of transformations, and was then expanded so as to

serve not only for a formulation of known interconnections, but

also as one of the most powerful and versatile research tools

for new interconnections which previously had not even been sus-

pected to exist. This will be shown in the following discussion

when the principles of Continuum Mechanics will be reviewed in

the light of the general theory of transformations.

B. The Dependence of Quantitative Observations on the Observed
   Object and on the Metrical System of Reference Acting as
   Observer

1. The Arbitrariness of the Scale Functions of the Metrical
   System of Reference and the Elimination of the Arbi-
   trariness by Way of Invariants and Co- and Contra-
   variants

Continuum Mechanics, as well as any other part of
Physics, can be developed into an exact science only through
quantitative observations, and the general theory of transfor-
mations starts from the recognition that every such observation
is essentially relative in the sense that it specifies some rela-
tion between the observed object and the observing system, and
hence varies for the same observed object with a change from
the arbitrary viewpoint of one observing system to that of
another. For quantitative observations the observing system
will have the form of a metrical system of reference, i.e.,
a set of measuring instruments with calibrated scales for all
the physical parameters under consideration, and the arbitrari-
ness of the viewpoint of the system will here appear as the
arbitrariness in the choice of the scale functions[1] according
to which the scales of the instruments are calibrated, i.e.,
the numerical values of the parameters are associated with the
divisions and subdivisions of the scales of the instruments.

---

1. The arbitrariness in the choice of the scale function is
well known and often used in graphical representations, when the
numerical values of a parameter may be distributed along the x
or y axis either linearly, or quadratically, or logarithmically,
or according to any scale function most suited to the particular
purpose in hand. It is, however, important to realize that the
same arbitrariness of the scale function is present in every
theoretical definition and experimental measurement of a parame-
ter, and for this reason the scale functions must here be regarded
as an integral part of every metrical system of reference.

In order to enable the metrical system of reference to measure all the physical parameters involved in Continuum Mechanics, it is important that the system be equipped not only with a calibrated coordinate system as a yardstick for the measurement of space, but also with a calibrated clock for the measurement of time, a calibrated balance for the measurement of mass, and calibrated gauges for the measurement of strains, stresses, etc. In any particular choice of the metrical system of reference all the scale functions involved in the various calibrations must be regarded as arbitrary since no laws of calibrations are known a priori.

In order to give physical significance to any interrelation of the numerical values of parameters, it is necessary to eliminate the arbitrariness in their scale functions, and this can be achieved by interrelating or combining the parameters under observation in various ways such that the description of the resultant interrelations, or combinations, have the simple _transformation properties_ of _invariance_, or of _co-_ and _contravariance_, against wider and ever wider groups of transformations from one arbitrarily chosen system to another. These invariants (or co- and contravariants) are of primary importance for the establishment of law and order free of arbitrariness (within a specified group of transformations), and the transformation theory, by spotlighting them, made it possible to derive in

Continuum Mechanics systematically all the various theoretical

concepts, laws and principles, as well as the design of all

experimental instrumentation from the said invariants or co-

and contravariants formed by appropriate combinations and inter-

reactions of the physical parameter involved. By this procedure

one achieved a considerable degree of control over the behaviour

of materials under mechanical actions, even in apparently para-

doxical cases, as, e.g., when normal pressures were generated

in laminar shearing movements.

In the procedure mentioned above the general theory

of transformations had to make use of a great number of develop-

ments of which some have already been well known in Classical

Physics, such as the Tensor Calculus, and the theories of

Mechanical Symmetry and Mechanical Similitude which will be

briefly discussed below in Sections 2 to 4 of this chapter.

Important though they were, one found these developments insuf-

ficient to provide effective control over the mechanical behav-

iour of materials, and therefore in need of supplementary devel-

opments, which were specifically designed for the purpose. They

will conveniently be discussed at a later stage, in connection

with the application of the general theory of transformation to

the principles of Continuum Mechanics (see sections 1 and 2 of

Chapter 3).

2. The Tensor Calculus as Key to the Goniometry

The Tensor Calculus was developed for the mathematical handling of physical parameters of all kinds which are distributed through space according to continuous analytical functions of the space coordinates, i.e., functions which can be developed in the neighborhood of any point in space into a Taylor series (convergent power series) with constant coefficients, proportional to the values of the local space derivatives of the functions.

The calculus made it possible to classify all the said physical parameters according to their transformation properties, and represent the distributions in space of the parameters by fields of tensors of various orders, with co- and/or contravariant components. This representation had the advantage of being free from the arbitrariness of the metrical system of reference[1] with regard to the scale functions of the space

1. The Tensor Calculus constituted only the first, but not the last step in the elimination of the arbitrariness in the metrical system of reference, because the arbitrariness was eliminated only with respect to the scale functions of the space coordinates, while it remained in the scale functions of the strains and stresses and other parameters measured by the system of reference. This remaining arbitrariness, and its elimination will be discussed in the Chapter C Sections 1 and 2, and in greater detail in Lectures VIII and X.

coordinates. A further advantage was derived from the above-
mentioned development into a Taylor series which showed that the
distribution of the values of a parameter throughout the whole
three-dimensional expansion of space was already fully determined
by the Goniometry, i.e., by the measurements taken at any one
point round the full solid angle of directions. Moreover, the
results of the Goniometry were already predetermined to a cer-
tain extent since the distribution of the values of the parameter
round the full solid angle of directions had to be the same as
that of a multilinear form of the space coordinates, the multi-
plicity of the form being determined by the order of the tensor
under consideration.

Full use will be made in this series of lectures of
the tensor calculus and in particular of the Goniometry, the
importance of which had not previously been noticed. It is the
Goniometry of the strains and stresses which has greatly simpli-
fied the theory of Continuum Mechanics as well as the design of
the instrumentation.

3. The Mechanical Symmetry

Another development which proved very helpful to the
application of the general theory of transformation to Continuum
Mechanics was derived from a consideration of the Mechanical
Symmetry, defined by the invariance of the description of

mechanical behavior under a group of symmetry transformations,
i.e., of transformations which are applied to the metrical
system of reference, and leave unchanged the distance between
any two points or, more exactly, the quadratic form by which
the distance between two points is expressed in terms of the
space coordinates of the metrical system of reference. A
complete systematic register of all conceivable different
classes of mechanical symmetry was then provided by the theory
of symmetry transformations, and the ordering into these classes
had the advantage that the materials collected in one class
had in their behavior certain general features in common.
In particular, one found that the symmetry class already deter-
mined the number of mutually independent components in every
mechanical property, and the elements in the symmetry class,
i.e., the axes and planes of symmetry, indicated the directions
along which each mechanical property reached an extreme value
(maximum or minimum). Moreover, one could elaborate the order
already achieved by the symmetry class through the application
of the representation theory. This theory associated with each
symmetry group a number of different representations by groups
of unitary matrices, and thus provided for the mechanical behav-
ior the means for a further ordering within each class of mechan-
ical symmetry. In this way one ordered, for instance, all
mechanical vibrations of a material or a given symmetry class

into different normal modes.[1]

4. The Mechanical Similitude

Yet another useful development was based on a study

of the _Mechanical Similitude_, defined by the invariance of the

description of the mechanical behavior under a group of simili-

tude transformations of the metrical system of reference, _i.e._,

under a group of linear homogeneous transformations of the units

of length, time and mass used in the metrical system of refer-

ence. The mechanical similitude served as a basis for dimension-

less analysis, and for the classical model techniques allowing

for changes of size, speed and specific gravity. A new generali-

zation of this model technique will be discussed below in Section C2.

C. The Principles of Continuum Mechanics in the Light of a General Theory of Transformations

1. Approximation through the Use of Newton-Galilei Systems of Reference

A discussion of the Principles of Continuum Mechanics

as seen in the light of a general theory of transformations can

be given here only in a first approximation in which one can

neglect the interactions between mechanical and electrodynamic

phenomena. Such interactions play an important role in the effects

---

1. The order established by the symmetry classes, as well as
that established by the normal modes applies equally well to all
electromagnetic phenomena, in particular to all spectra, what-
ever their origin, their frequency, etc.

described in Einstein's Special and General Relativity Mechanics and in the Heisenberg-Schroedinger-Dirac Wave and Quantum Mechanics, but become negligible when one limits the range of experimental conditions to movements with velocities which are small compared with the velocity of light, and to pieces of matter of sized which are large compared with the interatomic dimensions. It will be tacitly understood that the range of experimental conditions in Continuum Mechanics will be so limited, since it would far exceed the scope of this series of lectures to discuss the highly successful but very complicated application of the general theory of transformation to the formulation of the principles of Relativistic and of Wave and Quantum Mechanics.

Within the above specified range one can now make all quantitative observations in Continuum Mechanics with a metrical system of reference which uses the classical laws of the **Newton-Galilei Mechanics of Rigid Bodies** for calibrating the scales of all the measuring instruments. This calibration ensures that the measurements taken on all the various physical parameters involved fit into one self-consistent scheme in which the scale functions for time and mass are adjusted to those of the space coordinates according to the Galilei laws of the space time transformations, and Newton's law of the inertia of masses. Compliance to all these laws, and a tensor representation of all the physical

parameters involved, will reduce but not eliminate the arbitrariness in the metrical system of reference. In particular one can deduce the arbitrariness still remaining in the system from the lack of knowledge of any calibration laws which could specify the scale functions of strains and other dimensionless parameters, or determine the units of measurements for space, time and mass in the scale functions of the stresses, and other parameters expressed in terms of dimensions. The tensor representation does not help here, as, $\underline{e}.\underline{q}.$, one can measure the strain by any arbitrarily chosen scale function of the elongation tensor λ , such as $\lambda - I$, or $\lambda^2 - I$, or $\ln \lambda$ etc., where I denotes the unity tensor. It is important to recognize the remaining arbitrariness in the chosen metrical system of reference because it is just this arbitrariness which was removed for the strains and stresses by a new development of the general theory of transformations, resulting in the formation of scale invariant interrelations, as discussed below in Section 2 of this Chapter.

2. The Principles and Model Techniques in a Comprehensive Scheme of Continuum Mechanics

It was found convenient to study the principles and model techniques of Continuum Mechanics in the framework of a comprehensive scheme, providing three distinct sections, each governed by a different principle. The first section, Kinematics,

will be shown to be governed by the Principle of Preservation of Continuity (Lecture III and IV), the second section, Dynamics, by the Principle of Equilibrium of Forces[1] (Lecture V), and the third section, Mechanical Properties, by the Principle of an Equation of State (Lectures VI to X).

A discussion of the scheme based on the general theory of transformation showed that one could take over unchanged the classical formulations of the principles involved in the two first named sections, Kinematics and Dynamics, because one found that these formulations were already invariant against groups of transformations sufficiently wide to ensure general applicability within the full range of different materials and mechaniaal actions. However, in the third section dealing with the Mechanical Properties, the situation was different. Here one had to find a new formulation for the principle involved, because the classical formulation was invariant only within a narrow group of transformations, and had an accordingly restricted range of applicability.

A short report is given below of the results obtained in the various sections through the application of the transformation theory.

---

1. This principle is so formulated so as to include Newton's law of inertia by postulating an equilibrium between the inertia forces and the resultant of all the other forces.

In the first section, Kinematics, one deduced from the Principle of Preservation of Continuity certain invariants which served as a basis for a new model technique, in which a Trellis Mechanism was used to simulate the continuous movements of materials, and to provide solutions for boundary problems.

In the second section, Dynamics, a discussion of the transformation properties of the Principle of Equilibrium of Forces resulted in an appreciation of the different invariants which were the basis for the different conceptions of forces, introduced by Newton, Lagrange and Einstein respectively.

Finally, in dealing with the third section, Mechanical Properties, one had to recognize that these properties are defined by the interrelation of the strains and stresses which coexist in every differential cell of the material while the mechanical action is applied. It will be remembered that the classical theory had formulated such an interrelation by two linear laws, viz., Hooke's law of elasticity and Newton's law of viscosity, whose constant coefficients denoted the mechanical properties. Both these laws were severely limited as they took into account only strains of infinitesimally small amounts, and applied only to materials which were either "ideally elastic" or "ideally viscous." The insufficient generality of the classical formulations became increasingly troublesome with the discovery of an ever growing number of so-called "abnormal" mate-

rials, and "abnormal" phenomena of flow, the term "abnormal" being used here to express deviations of the mechanical properties from those mistakingly regarded as theoretically established. In due course one found abnormal behavior in most materials of technical or scientific importance, including metals, plastics, paints, rubbers, timber, colloidal gels and sols, etc., all of which had mechanical properties somewhere between ideally elastic and ideally viscous, and often exhibited changes of these properties dependent on the amounts of strains (large or small) contained in the pre-history of the applied mechanical actions. The increase in numbers and in importance of the deviations from the conventionally accepted classical laws finally made it imperative to replace the classical laws by a new principle with an Equation of State applicable to the mechanical properties exhibited in the behavior of materials of all kinds throughout the whole range of applied mechanical actions with strains of any amounts, large or small. The problem of formulating explicitly such a principle as an Equation of State was tackled by first adjusting to one another the scale functions of the strains and stresses that coexisted in every differential cell of the material while the mechanical action was applied, and then finding for these strains and stresses scale-invariant interrelations. The simplest such interrelation was referred to as "Anisotropic Similitude." It was defined for every differential

cell in a scale-invariant manner by postulating the coincidence of the directions in space along which the normal, tangential and cross-components of the strains and stresses extend with respect to planes of all orientations. Based on this scale-invariant interrelation one developed a new model technique of a general type which contained as a special case the classical model technique of mechanical similitude (mentioned earlier). Moreover, one proceded from the simple interrelation of Aniso-tropic Similitude to more complicated ones which were invariant to wider and ever wider groups of transformations, and proposed as the new principle for the mechanical properties an Equation of State, given by the particular scale invariant stress-strain interrelation which contained both the classical laws of elas-ticity and viscosity as special cases, and was invariant against the widest groups of transformations.

Experimental tests were made and showed that effective control over the behavior of materials of all kinds could be established over very wide ranges of experimental conditions, by taking as a guide not only the Principles of Continuity and of the Equilibrium of Forces, but also the newly proposed Princi-ple of an Equation of State and the new model techniques of the Trellis Mechanism and the Anisotropic Similitude.

It should be noted that it had been found convenient in the subsequent lectures to use some abbreviations viz. the

term a _system of reference_ will be used instead of metrical

system of reference, and the term _Galilei system_ instead of

Newton-Galilei system.

Seminar on Continuum Mechanics

Lecture III

Principles of Kinematics

Part I. The Preservation of Continuity

A. Introduction to the Kinematics of Continuous Media

   1. Systems of Reference and Kinematic Variables

   2. Continuity of Movements in Statistical Averages

B. The Preservation of Continuity

   1. Formulation of Principle

   2. The Kinematical Variables

   3. Conditions of Compatibility

   4. The Degrees of Freedom (Allowed and Forbidden) for
          Continuous Movements

   5. Goniometry of Kinematics

C. Theory of a Trellis Model

   1. Interpretation of Principle by Means of Lagrange
          Forces of Restraint

   2. Applications of Trellis Model

Seminar on Continuum Mechanics
(Lecture III)

Principles of Kinematics

Part I.   The Preservation of Continuity

A.   Introduction

   1.   Systems of Reference and Kinematic Variables

The Kinematics of Continuous Media is concerned with the theoretical description and experimental measurement of the kinematical variables, i.e., of variables which characterize the positions and movements of such media in space and time relative to some conveniently chosen system of reference.

There is a great deal of arbitrariness in the choice of the system of reference, as well as in that of the kinematic variables. In either case the choice will be made so as to fit in the best way the theoretical description to the experimental measurements. For different problems the choice will in general be different too, and in such cases one will have to change from one system of reference and one set of variables to another system and another set by appropriate coordinate transformations.

For the problems dealt with in this series of lectures it will be convenient to choose once and for all a Galilei system of reference which measures time with a mechanical clock and space with a rigid yardstick, and a system of coordinates which have all been calibrated in accordance with the Euclidean

3-2

principles of 3 dimensional geometry and Newton's principles of the inertia forces.* The kinematic can be described in terms of movements of either points or planes, and the kinematic variables are then so chosen that they express in the simplest manner the formation of the principle of preservation of continuity in the Galilei system of reference. In conformity to the general convention, the description in the chapter "Kinematics" will be given in terms of movements of points, but this description will have to be changed in the later chapter "Equation of State" into that of movements of planes.

The principles used for the calibration of the Galilei system of reference do not provide sufficient information for a description of the kinematics of continous media and the introduction of a further principle, namely, that of preservation of continuity, will be necessary. To see this one may note that the

---

* It will be noted that the Newtonian principles of the inertia forces have to be considered in the kinematics because otherwise the measurements of space and time would give results different from those obtained in a Galilei system. In relativistic mechanics, e.g., the electromagnetic clock would differ from the mechanical one, and the length of the rigid yardstick would vary with its movement.

principles used for calibration leave the movements of the points

of the medium mutually independent so that every point of the

medium would be allowed to move in 3 degrees of freedom arbitrar-

ily, and independently of the movements of the other points.

Considering that there is an infinitely large number, N say, of

points along an edge of a differential cell (however small) one

finds that there would be $3N^3$ degrees of freedom of movement

associated with every differential cell, and $3N^3 M^3$ such degrees

for any finite portion of the medium, when M denotes the infi-

nitely large number of differential cells along one dimension

of the said portion. If one had to deal with a real material

instead of the ideal continuous medium one would find the situa-

tion not much better because the number of actual particles

(nuclei and electrons) in the real material would still be un-

manageably large. In either case where would be little use at

this stage to introduce kinematical variables, since their vast

number (at least equal to the number of degrees of freedom in-

volved) would make it impracticable to specify each of these

variables individually, while on the other hand such an individ-

ual specification would be required as each variable is indepen-

dent of all the others. The way out of this impass is to apply

statistical methods over sufficiently large sections of space

and time.

2. Continuity of Movements in Statistical Averages

For a real material one derives statistical averages
on the basis of the assumptions and principles of thermodynamics
and of the kinetic theory of heat content, while for the ideal
continuous medium a corresponding assumption is made by the intro-
duction of a principle which serves the same purpose, and is re-
ferred to as the principle of preservation of continuity. The
name is not well chosen as the principle postulates much more
than the mere preservation of continuity. It postulates in fact
all that is necessary to develop the movement in time and space
into a convergent power series. However, the name will be re-
tained here in order to avoid the introduction of a new nomen-
clature. It will be seen from the following discussion that
this principle which is usually regarded as a purely mathemati-
cal formalism with little, if any, physical significance, is in
fact, dominating and controlling the main features of the whole
of Macroscopic Mechanic as described in the Kinematic Dynamic
and Mechanical Properties of Continuous Media. For application
of the Macroscopic Mechanics to the Microscopic one, it is, how-
ever, necessary to remember that the movement of a point of the
ideal continuous medium does not correspond to the movement of
any individual particle of the real material but to the linear
average of the movements of a large number of such particles
(e.g., ideal gas).

In this description the equations will be formulated either in terms of the invariant tensor notation or in terms of co- and contravariant components indicated by lower and upper indices respectively. All indices run from 1-3 and, as usual, summation will be implied over any index which occurs as upper and lower index in the same term. Either notation insures the invariance against all transformations of the coordinate system if applied to quantities which are true tensors, such as the coordinates of finite size,[1] and the volume capacities and densities of tensors, and in such cases the notation ensures invariance only within some restricted group of systems of coordinates in which the quantities transform in the same manner as true tensors. The Cartesian systems of coordinates form such a restricted group, and an equation will be marked with an asterisk whenever its invariance against coordinate transformation is restricted to the group of Cartesian systems.

---

1. See, e.g., Spain "Tensor Calculus" (Oliver and Boyd) p. 5. that space coordinates of infinitesimally small length, δx^k say, transform as contravariant vector components in all coordinate systems, while space coordinates X^k of finite length do not always, and transform as contravariant vector components only in systems of coordinates derived from a Cartesian system by linear transformations with constant coefficient.

B. The Preservation of Continuity

    1. Formulation of Principle

        For the formulation of the principle one considers in the movement of the medium two positions, an initial one, occupied at an instant of time $\overset{o}{t}$ and a terminal, occupied at an instant t.

        In order to identify the points of the medium one uses a system of coordinates which may be fixed in space. The points of the medium may then be labeled either in the initial position with the local space coordinates $\overset{o}{x}{}^{r}$, or in the terminal position with the corresponding coordinates x^{k} . This alternative leads to a description of the kinematic from two points of view, one referred to as the Lagrangian uses the $\overset{o}{x}{}^{r}$ as the independent and the x^{k} as the dependent coordinates, while the other referred to as Eulerian conversely uses the x^{k} as the independent and the $\overset{o}{x}{}^{r}$ as the dependent coordinates, according to the equations

$$x^{k} = F^{k}(\overset{o}{x}{}^{1}, \overset{o}{x}{}^{2}, \overset{o}{x}{}^{3}, t-\overset{o}{t}) \tag{1}$$

$$\overset{o}{x}{}^{r} = G^{r}(x^{1}, x^{2}, x^{3}, t-\overset{o}{t}) \tag{2}$$

        The movement, as described by equations (1) and (2) will henceforth be assumed to conform to the principle of preservation of continuity.

This principle postulates continuity, univalueness, and univalued reversibility not only for the functions F^k and G^r (which relate to one another the initial and terminal positions), but also for the time and space derivatives of these functions up to as high an order as may be required which is infinite for the problem under discussion. (The minimum requirement is for derivatives up to and including second order.)

Adopting the Lagrangian point of view one deduces from the definition of the principle of preservation of continuity that the functions F^k (and their derivatives) are regular analytical functions in time and space, i.e., functions to which Taylor's theory applies. According to this theory there is domain of finite size referred to as regular domain within which the functions F^k can be developed into convergent power series of time and space increments δt and $\delta \dot{x}^r$ round a center provided by a representative instant of time and particle in space, located within the said domain, viz.,

$$x^k + \delta x^k = F^k + \frac{1}{1!}\left[\delta \dot{x}^i \frac{\partial F^k}{\partial \dot{x}^i} + \delta t \frac{\partial F^k}{\partial t}\right] + \frac{1}{2!}\left[\cdots\cdots\right] \qquad (3)$$

where the Jacobi determinant of the first space derivatives must be different from zero, viz., $det\left\|\frac{\partial F^k}{\partial \dot{x}^i}\right\| \neq 0$ and > 0 because of the univalued reversibility of the movement and the exclusion of mirror inversion. For the purpose of kinematics, it suffices to formulate the principle by (3) but it may be noted that the

combination of (3) with the invariance of mass is an equation

which in the literature is described as Equation of Continuity.

From the Lagrangian point of view one can derive the

Eulerian (or vice versa) by exchanging the dependent and inde-

pendent space coordinates, and the functions F^k and G^r.

2. Kinematic Variables

The choice of kinematic variables is to a large extent

arbitrary and may be adjusted to the particular problem in hand.

Part I of the Kinematics deals with the problems connected with

the principle of preservation of continuity and the set of kine-

matic variables will be so chosen as to express this principle

in the simplest form. Part II of the Kinematics will deal with

problems connected with the resolution of movements into purely

rigid and purely deformatory components, and there one will ex-

press this resolution in terms of new sets of kinematic variables

which are defined as suitably chosen functions of the variables

introduced here. Finally, in the chapter dealing with the Equa-

tions of State the problems are concerned with the relations be-

tween the kinematic and Dynamic, and here again one has to intro-

duce new sets of kinematic variables and define them as suitably

chosen functions of the variables introduced in Part I and II.

Altogether one has here already a choice between a

variety of kinematic variables referring respectively to

(a) points or planes, with labels attached in either

(b) initial or terminal position (Lagrange or Euler) and with measures by any of a number of suitably chosen scale functions. There are many other alternatives of which some will be considered in Kinematics, Part II. Unfortunately, one has to consider all of them, since each one has the advantages and disadvantages for various problems.

Based on the considerations given above, one may now introduce kinematic variables with reference to equation (3) and it will suffice here to discuss these variables from the Lagrangian point of view as the discussion from the Eulerian point of view would be analoguous, mutatis mutandis, as explained above.

Definition

For the problems dealt with in Part I of the Kinematics one can define a local set of kinematic variables at any representative instant of time and particle in space by the successive derivatives in time and space of the functions F^k taken then and there. For the problem discussed in later chapters conventionally agreed functions of these derivatives may be used and the functions adapted to the particular problems in hand. For ease of reference the functions F^k will be regarded as derivatives of zero order, and thus included in the series of successive derivatives.

It will be noted that the local kinematic variables (being defined by the derivatives of the functions F^k) can be

determined within an infinitesimally small region round a repre-
sentative instant of time and particle in space, and once so
determined can be substituted in (3), and thereby completely
characterize the movement described by the functions F^k for
the regular domain of finite size, i.e., for some a finite inter-
val of time and some finite portion of the medium. The com-
pleteness of the characterization makes it possible to express
all sets of kinematic variables which may be introduced later,
as functions of the variables introduced here. It must be em-
phasized, however, that the characterization of the movement by
the local kinematic variables is given in an abbreviated form in
which it is understood that these variables will be used as
coefficient of the various powers of $\delta \dot{x}^r$ and δt, as indicated
in (3). Only in connection with the power series can the local
kinematic variable serve as a characterization of the movement.
It is then seen from the form of this series that the coefficient
of successive powers are tensors of increasing orders, and that
the local kinematic variables are identical with the components
of these tensors.

For an explicit calculation of the local kinematic
variables one has to decide on the position of the center of the
development (3) in relation to the system of reference. The
decision is arbitrary and there are two different conventions
in use. According to one convention the position of the center

is fixed, once and for all, within the regular domain of the development(3), while according to the other the center is left movable within that domain. From the two different conventions one deduces two different specifications of the local kinematic variables.

In the case of a fixed center one obtains constant values for all the tensors which appear in (3) as coefficients of the various powers, since all the tensor components are the time and space derivatives taken at the fixed center. Accordingly, all the local kinematic variables too will have here constant values and will form the components of tensors which are immovably attached to the fixed center.

In the case of a movable center one finds that the tensors which appear in (3) as coefficients of the various powers are now functions of the time and space coordinates of that center, since the tensor components are here the derivatives taken at a center whose local location varies in time and space. In this case all the local kinematic variables too will be functions of the time and space coordinates of the center and will here form the components of tensors which spread through the whole of the regular domain of the F^k and vary there with the location of the center in that domain.

For the characterization of the movement one may use the local kinematic variables with either convention, as the

results are equivalent and can be transformed into one another
by way of (3). Which convention to use will depend on the par-
ticular problem in hand, and often it will be advantageous to
change from one convention to another. In any case, it should
be clearly indicated which convention is used when, as other-
wise misunderstandings will arise.

3. Conditions of Compatibility

The necessity to distinguish clearly between the
kinematical variables attached to a fixed center, and those at-
tached to a moveable one can best be appreciated by considering
that the components of the former are mutually independent con-
stants, while those of the latter are mutually interrelated func-
tions of the (variable) coordinates of the center. The inter-
relations are known as Conditions of Compatibility and are de-
rived from a comparison of equation (1) with (3). Both equa-
tions describe the same movement in different ways. While (1)
uses in the description 3 and only 3 mutually independent func-
tions of the time and space coordinates, _viz._,

$$F^k \; (\overset{o}{x}{}^1, \; \overset{o}{x}{}^2, \; \overset{o}{x}{}^3, \; t - \overset{o}{t}) \quad \text{for } k = 1, 2, 3$$

equation (3) uses an infinite number of them, _viz._, all the local
kinematic variables $F^k(\overset{o}{x} \; t)$, $\dot{F}^k(\overset{o}{x} \; t)$, etc. It follows that
of all the local kinematic variables given as functions of time
and space coordinates only three can be mutually independent,
namely, three $F^k(\overset{o}{x} \; t)$ for $k = 1, 2, 3$, while any four or more

must be interrelated. The algebraic form of the interrelation will depend on the particular kinematical variables chosen. As an example one may consider the nine variables F^k_m ($\overset{o}{x}$ t) (for k = 1, 2, 3 and m = 1, 2, 3) which can be derived as the components of the gradient of a vector F ($\overset{o}{x}$ t), viz.,

$$\nabla F\ (\overset{o}{x}\ t)\ =\ F^k_m\ (\overset{o}{x}\ t)\ e_k\ e^m \tag{4}$$

where e_k and e^m are the co- and contravariant unit vectors. Remembering then the identity

$$\nabla \times \nabla \ \equiv\ 0 \tag{4a}$$

one obtains from (4) and (4a) the compatibility condition

$$\nabla \times \nabla\ F(\overset{o}{x}\ t)\ =\left[\frac{\partial}{\partial \overset{..}{x}^a}F^{a+1}\ (\overset{o}{x}\ t)\ -\ \frac{\partial}{\partial \overset{..}{x}^{a+1}}F^a_m\ (\overset{o}{x}\ t)\right] e_{a+2}\ =\ 0 \tag{4b}$$

The compatibility conditions are always trivially fulfilled if one derives the kinematic variables by successive gradient operations as in (4). However, the conditions are not trivial and, in fact, most important if one were to choose the nine functions F^k_m ($\overset{o}{x}$ t) etc. arbitrarily, or tried to impose on them certain conditions, such as the propagation of harmonic waves, or volume changes corresponding to a prescribed temperature gradient, etc.

4. The Degrees of Freedom (Allowed and Forbidden) for Continuous Movements

For an assessment of the degrees of freedom involved in the movements allowed by the principle of preservation of continuity one may conveniently use the local kinematic variables together

with the convention of a fixed center for the developments into power series as in (2.2). Moreover, one need not consider the development in time, as this affects only the velocities, accelerations, etc., of the movement but not the freedom in space of its path. Accordingly, one may assume $\delta t = 0$ and then find from (3)

$$x^k + \delta x^k = F^k + \frac{\delta \overset{o}{x}^l}{1!} F^k_l + \frac{\delta \overset{o}{x}^l \delta \overset{o}{x}^m}{2!} F^k_{lm} + \ldots \qquad (5)$$

where all the F's are mutually independent constants so that each nonvanishing term will specify one degree of freedom of movement, and the degree of the term will indicate the order of magnitude of that movement, when it is agreed that the $\delta \overset{o}{x}$ are infinitesimally small of first order. By evaluating (5) in successive steps of approximation of order 1, 2 ... one can cross out in (5) all terms which contain products of more than 1, 2 ... n of the $\delta \overset{o}{x}$. One can then find the number of degrees of freedom involved in that approximation by counting the number of the remaining terms. For the most general class of movements compatible with the principle of preservation of continuity all the F's with various indices may be different from zero, and one finds here for approximations of orders 1, 2 .... n the number of degrees of freedom equal to 12, 39 and $3 \frac{3^{n+1}-1}{2}$ respectively. In many cases the movements with which one has to deal are not in the most general class, but in a more special one, in which some of the F s may be zero. In such cases the numbers of degrees of freedom are found

by first discarding in (5) all the terms which are multiplied by an F of zero value, and then proceeding in successive approximations as indicated above. One then finds for an infinitesimally small region that all the movements, whether general or special, will have a number of degrees of freedom which is given in approximation of order 1, 2 ... n by

$$d \leqslant \begin{cases} 12 & \text{for first approximation} \\ 39 & \text{for second approximation} \\ 3 \ \dfrac{3^{n+1}-1}{2} & \text{for } n^{th} \text{ approximation} \end{cases} \tag{6}$$

When one procedes from the infinitesimally small region surrounding the center to regions of finite size within the regular domain of the F^k one finds that no new degrees of freedom come in as the development (5) hold true for the whole regular domain. The number of degrees of freedom involved will depend on the class of movement considered, and on the degree of accuracy required. For the most general class of movement the obtained degree of accuracy will decrease the further one goes away from the center of the development, but even here only a limited number of degrees of freedom will be involved as the postulated convergence of the power series ensures that any required degree of accuracy can be obtained from a limited number of terms in the series. For special classes of movements the situation is more advantageous and it should be particularly noted that there is an important class of movements known as "linear" or "homogeneous" for which perfect accuracy is obtained, no matter how far one departs from the center

of the development, by taking account only of the terms of zero
and first order in the development, and thus involving only 12
(or less) degrees of freedom for any region large or small.

It follows from the above discussion of the degrees of
freedom that the principle of preservation of continuity controls
the kinematic of continuous media very tightly. Considering, e.g.,
the kinematic of a differential cell of the medium it has been
shown in the introduction that in the absence of the principle
there are N^3 degrees of freedom involved (where N is an infinitely
large number), while in the presence of the principle this number
has been reduced to 12 in a first approximation. In other words
the principle determines for all media, and under all conditions
the movements in N^3 degrees of freedom in a first approximation
by postulating that all these movements must be zero in the said
approximation. Moreover, in the remaining 12 degrees of freedom
the changes in the movements from one cell to the neighboring
cells are restricted by the conditions of compatibility by which
the principle controls the changes in the movements which may occur
from one differential cell to the adjacent ones. There is a simple
geometrical interpretation of the compatibility conditions which
may be derived as follows.

Let a three dimensional differentially fine grid be im-
printed in the medium in the initial position, and let one of the
intersection points be marked together with the eight differential

cells which meet there, and together fill completely the full solid angle round the point. By the movement to the terminal position the various cells will, in general, suffer different distortions but must retain their contacts with one another and with the marked point (because of the preservation of continuity), so that together they still fill the full solid angle round the point. Hence when the solid angle of one cell decreases, the corresponding angle of one of the adjacent cells must increase, so that the sum of the solid angles of the eight cells always adds up to a full solid angle in space. This is the geometrical interpretation of the Compatibility Condition.

5. Goniometry of Kinematics

The study of the kinematics of continuous media is greatly simplified by the application of the principle of the preservation of continuity. Instead of studying the movement over any finite intervals of time and throughout any finite portions of a medium, it will suffice to study the Goniometry, i.e., the variation of the movement in the infinitesimally small neighborhood of an instant of time, and of a point of the medium round the full solid angle of directions. It is from this Goniometry that one determines at the chosen instant of time and point in the medium the local kinematic variables as the successive time and space derivatives of the movement then and there. Once these derivatives are determined in sufficiently high orders, they can

be used as coefficients of a convergent power series which will
then specify the movement of the medium throughout the regular
domain of time and space in which the power series converges.

Moreover, even within the Goniometry of the infinitely
small neighborhood of an instant of time and point in the medium,
the principle does not allow arbitrary variations of the move-
ments along different directions in space. Instead it allows
to predict the variations round the full solid angle of direc-
tions by showing for the development of the movement into a con-
vergent power series that the coefficients of this series, i.e.,
the local kinematic variables are tensors of various orders,
and must therefore vary round the full solid angle as linear
functions of the direction cosines associated with any one of
the indices of the tensors. This prediction is all the more
useful as the linear region covers the whole finite range of the
two angles which specify the directions round the full solid
angle.

The application of the principle to a determination of
the whole of the kinematics by way of the Goniometry, as described
above, is of great interest both to the theory and to the experi-
ments. For the theory it may be noted that nothing can interfere
with the linearity of the region at a point round the full solid
angle of directions, or the interconnection between the movement
there in the neighboring infinitesimally small elements of time

and space, and the movement during finite intervals of time and finite positions of the medium, no matter what forces of gravity, inertia, viscosity and elasticity may be applied to the medium. Hence the precalculation of the kinematics from the Goniometry will be valid whatever the conditions may be under which the medium is observed. For the experiments the instruments for testing the kinematics can be so designed to take advantage of the simplifications introduced by the Goniometry. In particular, it will suffice to measure the movement at any instant of time and point in the medium across three mutually perpendicular planes in order to calculate it round the full solid angle of directionally linear interpolation.

C. Theory of a Trellis Model

    1. Interpretation of Principle of Preservation of Contin-
          uity by Means of Lagrange Forces of Restraint

        A physical interpretation of the principle can be given by means of a theorem introduced by Lagrange, in which he showed how one can replace any geometric restriction imposed on freely moving particles by appropriately chosen forces of restraint. Applying this theorem to the conditions imposed by the principle, one can find the corresponding Lagrange forces of restraint, and construct from them a mechanism which will guide the points of the medium along the movements allowed by the principle and prevent

them from executing movements that are forbidden.

It will be convenient to formulate the geometric conditions imposed by the principle in terms of invariants. The corresponding Lagrange forces can then be determined as infinitely strong forces which rigidly interconnect the points along the said invariants, and so "prevent" the invariants from changing. The rigid interconnection then forms the backbone of the required mechanism.

As explained in the preceding section it will be sufficient to consider the infinitesimally small neighborhood of a point. In order to find there the invariants one may imprint in the medium a network of infinitesimally small cells, formed by a grid of three families of intersecting lines and surfaces, and one may then observe the changes occurring in the grid while the medium moves from the initial to the terminal position. According to the principle there will be no "diffusion" of points of the medium from one cell of the grid to another, and points which lie on a line or surface of the grid in a certain order, or which lie in the intersection of two or more lines or surfaces, will remain on that line or surface in the same order, or will remain in the said intersections respectively, however much the grid may have been distorted. Moreover, in a first approximation parallel and equidistant straight lines, or plane surfaces, will remain parallel, equidistant and straight, or plane, respectively,

while in a second approximation one has to take deviations into account in accordance with the conditions of compatibility to which the adjacent differentially small cells of the grid are subjected. Finally, considering in the grid the distances between neighboring points along lines of all directions round the full solid angle, one finds in general elongations along some lines, contractions along others, and lines of zero elongations in directions between the two.[1] The pattern of lines of zero elongation will suffer angular changes from the initial to the terminal position but will have as invariants the points of intersections and their distances apart. Introducing then into the pattern of lines of zero elongations infinitely strong Lagrange forces of restraint as rigid interconnections in accordance with the said invariants one obtains a mechanical model. In this model pinpointed universal joints will replace the points of intersection, and straight rigid rods along the lines

1. In the special cases in which one observes along all directions, either elongations only, or contractions only, the lines of zero elongation cannot be found in real space but they can still be determined in complex space so that the formal algebraic procedure will not be affected. The said special cases are comparatively rare since most real materials are almost incompressible so that elongations in some directions will be compensated by contractions in others.

of zero elongation will bridge the distances between consecutive points on these lines. The whole mechanism incorporates all the necessary and sufficient requirements of the principle, as it offers no resistance against all the angular changes "allowed" by the principle, but offers infinitely strong resistance against all the "forbidden" changes, i.e., against all changes of the invariants. In particular, the forbidden changes in the points of intersection, and in the distances between consecutive points along the lines of zero elongation are resisted by the infinitely strong forces corresponding to the pinpointing of the pivots, and the rigidity (inextensibility and inflexibility) of the connecting rods. The model will be referred to as a "Trellis Model" since it operates in the simple case of a homogeneous two-dimensional movement like the well-known type of collapsible garden trellis with two intersecting series of rigid rods pivoted to one another where they cross, and capable of expanding along some directions while contracting at the same time along others. The Trellis Model can be constructed between any two positions of the medium (arbitrarily referred to as the initial and terminal position) but the model so constructed will not, in general, fit the intermediate positions.[1]

---

1. This is because the lines of zero elongation have different directions in the medium when constructed for different pairs of initial and terminal positions.

2. Applications of the Trellis Model

There have been many useful applications of the Trellis Model, such as the predetermination of sliplines in fatigue experiments, the solution of complicated boundary problems for anisotropic sheets of materials (in particular, parachute fabrics, and other woven and knitted fabrics), the construction of a new type of stretching device, a simple geometrical interpretation of the conditions of compatibility, etc. However, there is no need to burden this report with a description of the various applications since such a description has already been published elsewhere.

It will suffice to state here that one can determine in a quantitative manner the lines of zero elongation, and hence the Trellis Model, for any pair of positions of the medium by constructing (or calculating) at every point of the medium the two concentric strain quadrics (known as the strain ellipsoid and the reciprocal strain ellipsoid) of which one changes its shape from a unit sphere into an ellipsoid, while the other changes conversely from an ellipsoid into a unit sphere. The lines of zero elongation are then found in either position along the cone of diameters associated with the intersection of the two quadrics.

Seminar on Continuum Mechanics

Lecture IV

Principles of Kinematics

Part II. Superposition of Movements and
Resolution into Rigid and Deformatory Components

A. Introduction

   1. Generalization of Classical Theory of Infinitesimally
         Small Deformations

   2. Application of Group Theory

B. The Superposition and Resolution of Movements

   1. The Laws of Superposition

   2. The Laws of kesolution

C. Discussion

   1. General Movements

   2. Special Movements

D. The Kinematic Variables

E. Differential and Integral Calculus of Movements

Seminar on Continuum Mechanics

(Lecture IV)

Principles of Kinematics

Part II. Superposition of Movements and
Resolution into Rigid and Deformatory Components

A. Introduction

1. Generalization of Classical Theory of Infinitesimally

Small Deformations

The movements compatible with the principle of preser-
vation of continuity have been described so far relative to a
chosen Galilei system of reference in terms of kinematical vari-
ables defined as the time and space derivatives of the functions
which relate to one another the coordinates of the initial and
terminal position of the medium.

The Galilei system of reference fixed in space (as in-
dicated in Part I) will be retained, but will be used now for a
description in terms of kinematic variables corresponding to the
purely rigid and purely deformatory components of the movements.
Such a description together with a definition of the "pure" com-
ponents had already been given in the classical theories but was
restricted to infinitesimally small deformations. Here this re-
striction will be discarded, and a description will be given
which applies to deformations of all sizes, large or small, in
movements which comply with the principle of preservation of con-
tinuity.

In dealing with deformations of all sizes many new problems arise both for the superposition of movements as well as for their resolution into purely rigid and purely deformation components. These problems have to be well understood in order to account for and even predict, theoretically the many new phenomena which one encounters in experiments, and which are unexpected from the classical point of view, and sometimes have a paradoxical appearance. The said phenomena are, of course, already present in the case of infinitesimally small deformations but they are then infinitesimally small of second or higher degree, and thus negligible, while for large deformations they are predominant, as then the effects of first degree become negligibly small compared with those of second and higher degrees.

2. The Application of Group Theory

Group theoretical considerations will here be of fundamental importance since the theoretical and experimental handling of the movements, and of their resolution into purely rigid and purely deformatory components will depend on the group properties of the movements and components. It will be seen in particular that according to whether successive movements and components do, or do not form a commutative group one can, or cannot, characterize the movements by kinematic variables which are additive, since addition is always commutative, and thus incompatible with a noncommutative group.

Additive kinematic variables lend themselves easily to the application of differential and integral calculus, as the sum of the kinematic variables associated with the individual movements in succession will then characterize the movement resulting from the succession. Nonadditive kinematic variables have a nonadditive superposition law, and these require that great care be taken with the application of the differential and integral calculus, as in this case the characteristics of the movement during a differential interval of time will not be identical with the differences between the characteristics at the beginning and end of the interval, nor will the characteristic for a finite interval of time be identical with the integral over the characteristics associated with the movement during a differential interval of time.

It will suffice here again (as in Part I) to discuss explicitly only the Lagrangian point of view, as one can deduce from it the Eulerian one, or any other point of view, by appropriate transformations of the variables and functions used.

For ease of discussion it will often be convenient to use a system of reference fixed in a point of the medium instead of being fixed in a point of space. This will eliminate the purely rigid translations of the medium, which are irrelevant for most of the problems here considered. Whenever required, the translations are reintroduced by the application of an appropriate transformation of the system of reference.

B. The Superposition and Resolution of Movements

    1. The Laws of Superposition

        As all the movements and their components are assumed to satisfy the principle of preservation of continuity, one can characterize them throughout finite intervals of time, and for finite portions of the medium by (as in Part I) the goniometry of the local kinematic variables measured in the infinitesimally small neighborhood of any instant of time, and point in space round the full solid angle of directions. The local kinematic variables are again tensors of various orders, and it will suffice here to consider them in the interior of a differential cell and to describe them there as homogeneous, i.e., independent of the space coordinates. (If and when necessary, the tensors can be extended through time and space, and developed into a convergent power series of the time and space coordinates and so provide a characterization of an approximation of second and higher degree.) Introducing then a system of reference fixed in the point of medium considered (thereby eliminating the purely rigid translation), the movement described in terms of Lagrangian variables will appear in the form

$$\delta x = \delta \overset{\circ}{x}. \qquad \text{where} \qquad (1)$$

$$\psi = \begin{cases} \nabla \delta x \\ \dfrac{\partial F^k}{\partial \overset{\circ}{x}_m}\, e_k e^m \\ F^k_m\, e_k e^m \end{cases} \qquad (2)$$

For an investigation of the superposition of an ordered succession of the local movements at a point of the medium, one considers a differential cell surrounding the point in three positions, (0), (1), and ($\bar{1}$) say. Using then for the positions (0) and (1) the same notation as in Part I, and denoting by a bar the quantities referring to the position ($\bar{1}$) one finds for the local movements (01) (1$\bar{1}$) and (0$\bar{1}$)

$$\int x = \int^o_x . \Psi \tag{3}$$

$$\int \bar{x} = \int x . \Upsilon \tag{4}$$

$$\int \bar{x} = \int^o_x . \bar{\Psi} \tag{5}$$

where the ordered succession of the local movements (01) and (1$\bar{1}$) produces by superposition a resultant local movement (0$\bar{1}$) for which the superposition law is derived from (3) (4) and (5) as

$$\bar{\Psi} = \Psi . \Upsilon \tag{6}$$

One deduces from (6) that all local movements form by superposition a group (i.e., form an assembly which satisfies the four conditions necessary and sufficient for a group) but this group is not commutative, as the superposition of local movements according to (6) is by tensor dot multiplication, and this is not, in general, independent of the order of succession.

One may not reintroduce the system of coordinates fixed in a point in space, and so take into account the translations of the medium. For the superposition law it can then be shown

that the translations in the successive local movements add up to the translation of the resultant local movement.

While the laws of superposition are comparatively simple, the laws of resolution are of considerable complexity.

2. The Laws of Resolution

An arbitrarily given movement will in general be a composite one, involving not only a change in the outer location of the medium, i.e., relative to a system of reference fixed in space, but also in the inner location, i.e., in the relative positions of the points of the medium to one another. While the changes in the outer location are independent of the internal structure and associated with a rigid movement, one finds that the changes in the inner location will be associated with a deformatory movement, and will bring into play all the complexities of the internal structure. It is for this reason that any given composite local movement will have to be resolved into a purely rigid and a purely deformatory component.

It will henceforth be understood (unless explicitly stated otherwise) that all movements and their components, as well as the associated kinematic variables will be local ones, i.e., related to the neighborhood of the point considered, and for ease of reference the term local may then be omitted.

Because of the group formation, it is possible to carry out the superposition of the movements, as well as the resolution

into the purely rigid and purely deformatory components within
the framework of the Goniometry of the differentially small cell
surrounding the point of the medium considered. While throughout
all the movements (01), $(1\bar{1})$ and $(0\bar{1})$, the distances between the
points of the medium in the differential cell remain infinitesi-
mally small, one finds for the said movements that the associated
changes of directions, and percent elongations are of any amounts,
large or small, so that the purely rigid and purely deformatory
component will have unlimited amounts of rotation and of deformation

In order to define algebraically the purely rigid and
purely deformatory movements, one considers the changes in length
which are suffered during the movements by the vectors irradiat-
ing in all directions from the point of the medium considered.
One finds that the ratio of the squares of the length of the
vectors in the movement (01) is given by

$$\frac{\delta x \cdot \delta x}{\delta \dot{x} \cdot \delta \dot{x}} = \frac{\delta \dot{x} \cdot \Psi \cdot \widetilde{\Psi} \cdot \delta \dot{x}}{\delta \dot{x} \cdot \delta \dot{x}} \tag{7}$$

where $\Psi \cdot \widetilde{\Psi}$ is symmetric according

$$\widetilde{\Psi \cdot \widetilde{\Psi}} = \widetilde{\Psi} \cdot \widetilde{\Psi} = \Psi \cdot \widetilde{\Psi} \tag{8}$$

Now let Ψ be written as Ω or Θ according to whether the
movement is purely rigid or purely deformatory.

One then finds for the purely rigid movement that its
representative tensor Ω must have orthogonal symmetry since
for a rigid movement the ratio in (7) must be equal to unity, viz.

$$\Omega \cdot \widetilde{\Omega} = I \tag{9}$$

and one finds for the purely deformatory movement that its representative tensor Θ must be symmetric,[1] according to

$$\Theta \cdot \tilde{\Theta} = + \left(\Psi \cdot \tilde{\Psi} \right)^{\frac{1}{2}} \tag{10}$$

One can calculate the main values and main axes of the tensors Ω and Θ from the secular equation

$$T_1 e \;=\; T_1' e \quad \text{(for } T = \begin{cases} \Omega & \text{or} \\ \Theta \end{cases} \text{)}$$

where T' is one main value, and e a unit vector in the direction of the associated main axis. The result can then be

_____

1. To prove this one writes first formally Θ as a product of the symmetrical tensor $\left(\Psi \cdot \tilde{\Psi} \right)^{\frac{1}{2}}$ and an unknown tensor ν viz:

$$\Theta = \left(\Psi \cdot \tilde{\Psi} \right)^{\frac{1}{2}} \cdot \nu \tag{11}$$

One then finds from (11)

$$\Theta \cdot \tilde{\Theta} = \left(\Psi \cdot \tilde{\Psi} \right)^{\frac{1}{2}} \cdot \nu \cdot \tilde{\nu} \cdot \left(\Psi \cdot \tilde{\Psi} \right)^{\frac{1}{2}} \tag{12}$$

and hence

$$\nu \cdot \tilde{\nu} = I \tag{13}$$

which means that ν must be a tensor of orthogonal symmetry, and thus represent a rotation. In order that Θ should be a purely deformatory movement one will then postulate that in (11) should be unity, so that

$$\Theta = + \left(\Psi \cdot \tilde{\Psi} \right)^{\frac{1}{2}} \qquad \qquad \underline{q}.\underline{e}.\underline{d}. \tag{14}$$

where the + sign is chosen because Θ represents an elongation ratio and is therefore essentially positive.

interpreted geometrically as follows:

The purely rigid movement, or what amounts to the same, its representative orthogonal tensor Ω acts on all the vectors $\delta\overset{0}{x}$ in the position (0) as an operator which rotates them all by the same angle round a certain axis of rotation. To calculate the angle and axes of rotation one notes that an orthogonal tensor has three main axes with main values, 1, $e^{i\alpha}$ and $e^{-i\alpha}$ where α is the angle of rotation. Of the three main axes only one is real, and this corresponds to the main value 1 and points along the direction of the axis of rotation.

The purely deformatory movement or, what amounts to the same, its representative symmetrical tensor Θ acts on all the vectors $\delta\overset{0}{x}$ as an operator which subjects them all to the same elongations (or contractions) along certain three mutually perpendicular directions. To calculate said elongations and directions one notes that a symmetrical tensor has always three real values and three real mutually perpendicular main axes, and these determine respectively the amounts and the directions of elongation (or contraction) known as the main elongation ratios λ and the main axes, <u>viz</u>.,

$$\Theta_i = \lambda_i \quad \text{(for } i = 1, 2, 3) \tag{15}$$

It should be noted that the movements defined above as purely rigid and purely deformatory are mutually exclusive only insofar as they refer to the whole collection of vectors

$\oint_{\delta}^{o}x$ irradiating from the point considered. With regard to this
collection the pure rigid rotation does not involve any deforma-
tion, and the purely deformatory movement does not involve any
rotation. However, this mutual exclusiveness does not apply
to any single vector $\oint_{\delta}^{o}x$. For such a single vector the purely
rigid movement does not involve a deformation but the purely
deformatory movement involves in general a rotation as well as
an elongation, and this rotation vanishes only for the main axes,
and for the linear average, or the resultant of the rotations
suffered by the individual vectors of the whole collection.

In the definitions given above, the coordinate system
had been fixed in a point of the medium. Proceeding then to a
system fixed in a point in space one has to take account of the
translations. These are always purely rigid so that the most
general rigid movement will be a translation combined with a
rotation.

Having defined separately the purely rigid and the
purely deformatory movement in terms of algebraic equations,
one can now carry out the resolution of any given continuous
movement into an ordered succession of two component movements,
one purely rigid, and the other purely deformatory. The two
components will in general not commute, and one will therefore
have to consider an alternative between two resolutions (or
superpositions) according to which of the components is anterior

and which posterior in the succession. Both resolutions and
superpositions will have physical significance, and will there-
fore have to be specified separately. Accordingly, one finds
for the movements (01) (1$\bar{1}$) and (0$\bar{1}$)

$$\psi = \begin{cases} \Omega_a \cdot \Theta_p \\ \Theta_a \cdot \Omega_p \end{cases} \tag{16}$$

$$\psi = \begin{cases} \omega_a \cdot \vartheta_p \\ \vartheta_a \cdot \omega_p \end{cases} \tag{17}$$

$$\bar{\psi} = \begin{cases} \bar{\Omega}_a \cdot \bar{\Theta}_p \\ \bar{\Theta}_a \cdot \bar{\Omega}_p \end{cases} \tag{18}$$

where the index   a   or   p   refers to the anterior or posterior
position in the sequence. One then deduces from (16) for the
movement   (01)

$$\Omega = \Omega_a = \Omega_p = \psi \cdot \left[+ \left(\tilde{\psi} \cdot \psi \right)^{-\frac{1}{2}} \right] = + \left(\psi \cdot \tilde{\psi} \right)^{-\frac{1}{2}} \cdot \psi \tag{19}$$

$$\Theta_a = + \left(\psi \cdot \tilde{\psi} \right)^{\frac{1}{2}} \tag{20}$$

$$\Theta_p = + \left(\tilde{\psi} \cdot \psi \right)^{\frac{1}{2}} = \Omega^{-1} \cdot \Theta_a \cdot \Omega \tag{21}$$

This means that the purely rigid component is uniquely defined
in (19) by the orthogonal tensor Ω, while the purely deforma-
tory component will be given by (20) and (21) and will be equal
to either Θ_a or Θ_p according to whether it is anterior or
posterior to the rotation. The two tensors Θ_a and Θ_p differ
from one another only by their orientation so that the anterior
one can be transformed into the posterior one by the application

of the rotation Ω .[1] A resolution of the movements (1Ī). and (0Ī) may be carried out in terms of Ψ and $\widetilde{\Psi}$ and yields results analogous to (19), (20), and (21).

The equations (19) to (21) introduce new sets of kinematic variables Ω , Θ_a , Θ_p , which are defined as functions of the variables Ψ defined in Part I. The new variables will serve the purpose of specifying separately the purely rigid and purely deformatory components in the movement (01). Using then equations analogous to (19) to (21) for the resolution of the movements (1Ī) and (0Ī) one can now formulate the superposition law (6) in the new kinematic variables as

$$\bar{\Omega} = \Omega \cdot \Theta_p \cdot \vartheta_a \cdot \omega \cdot \left(\omega^{-1} \cdot \vartheta_a \cdot \Theta_p^2 \cdot \vartheta_a \cdot \omega \right)^{-1/2} \tag{22}$$

$$\bar{\Theta}_a = \Omega \cdot \left(\Theta_p \cdot \vartheta_a^2 \cdot \Theta_p \right)^{1/2} \cdot \Omega^{-1} \tag{23}$$

$$\Theta_p = \omega^{-1} \cdot \left(\vartheta_a \cdot \Theta_p^2 \cdot \vartheta_a \right) \cdot \omega \tag{24}$$

---

1. It may be noted that the main axes of Θ_a mark in the initial position the directions of lines in the medium which are going to suffer the extreme elongations (or contractions) through the subsequent movement; similarly the main axes of Θ_p mark there the directions which the same lines occupy in the terminal position, and the tensor Ω represents the rotation of the axes of Θ_a into those of Θ_p . One can then deduce the main values of Θ_a and Θ_p by comparing (15) with (21) and (22), and finds

$$\left(\Theta_a \right)_{(i)} = \left(\Theta_p \right)_{(i)} = \Theta_{(i)} = \lambda_{(i)} \qquad \text{(for } i = 1, 2, 3\text{)}$$

Equations (19) to (21) and (22) to (24) give respectively the laws or resolution into, and superposition from purely rigid and purely deformatory components.

The resolution of the movements into purely rigid and purely deformatory components makes it necessary to resolve in a like manner the conditions of compatibility which were dealt with in Part I of the kinematic for unresolved movements only. The conditions are given here only for the purely deformatory components because these are of particular practical importance. The conditions can be formulated with the help of a differential operator, known as the Cauchy-Rieman Curvature tensor $\mathcal{R}$ say. This tensor is of fourth order and must vanish[1] according to

$$\mathcal{R}(g) = 0 \qquad \text{for} \quad g = \begin{cases} \Theta_a^i \\ \Theta_p^i \end{cases} \tag{25}$$

where

$$\mathcal{R}^{l}_{.jnp} = \frac{\partial}{\partial \bar{x}^n}\left\{\begin{matrix} l \\ j\,p \end{matrix}\right\} - \frac{\partial}{\partial \bar{x}^p}\left\{\begin{matrix} l \\ j\,n \end{matrix}\right\} + \left\{\begin{matrix} l \\ n\,s \end{matrix}\right\}\left\{\begin{matrix} s \\ j\,p \end{matrix}\right\} - \left\{\begin{matrix} l \\ p\,s \end{matrix}\right\}\left\{\begin{matrix} s \\ j\,k \end{matrix}\right\} \tag{26}$$

and the $\{\ \}$ indicate the Christoffel differential operators of second kind. The geometrical meaning is again the one discussed in Part I, and illustrated by a Trellis model of adjoining differential cells.

---

1. For proof see Levi Civita, "The Absolute Differential Calculus," p. 198 (Blacky 1947).

C. Discussion

1. General Movements

In a given general movement, the medium moves along a certain path through an ordered series of positions, (0), (1), ...(n)..., ($\bar{1}$) say. Starting from an arbitrarily chosen initial position (0), one can imprint in the medium a triad of mutually perpendicular directions along which the medium suffers the maximum elongations (or contractions) between the positions (0) and ($\bar{1}$). This trirectangular triad is fixed in the medium but changes its orientation in space from that of the main axes of $\bar{\Theta}_a$ in the position (0) to that of the main axes of $\bar{\Theta}_p$ in the position ($\bar{1}$). There is an infinite variety of different paths by which the medium could move from the position (0) to ($\bar{1}$), and among these one can always find a privileged one, with positions (0), (1)*...(C)*...(n)*...,($\bar{1}$) say, such that the trirectangular triad imprinted in the medium remains trirectangular in all the positions of the privileged path. The algebraic condition for the privileged path is that the tensor Θ_p^* and ϑ_a^* should be coaxial in a system of reference fixed in space, and one then finds from (22) to (24)

$$\bar{\Omega} = \Omega^* \cdot \omega^* \qquad (27)$$

$$\bar{\Theta}_a = \Omega^* \cdot \left(\Theta_p^* \cdot \vartheta_a^* \right) \cdot \Omega^{*-1} = \Theta_a^* \cdot \left(\Omega^* \cdot \vartheta_a \cdot \Omega^{*-1} \right) \qquad (28)$$

$$\Theta_p = \omega^{*-1} \left(\vartheta_a^* \cdot \Theta_p^* \right) \cdot \omega^* = \vartheta_p^* \cdot \left(\omega^{*-1} \cdot \Theta_p \cdot \omega^* \right) \qquad (29)$$

where the quantities with a star refer to any of the movements
(Oc*) and (c*T) along the privileged path. In this case
one finds that there is coaxiality in the medium (but not in
space) of all the tensors of the pure deformations and of their
time derivatives (deformation velocities, accelerations, etc.)
so that one can introduce a system of reference with axes fixed
in the medium along the imprinted trial of trirectangular direc-
tions such that all the said tensors are simultaneously in
diagonal form, i.e., on their main axes.

The actual path of the medium will, in general, be
different from the privileged one (as, e.g., in laminar flow).
Then the tensors Θ_ρ and ϑ_a will not be coaxial in the
system of reference fixed in space, and in that case it will
be impossible to find any trirectangular triad which can be
imprinted in the medium and remains trirectangular in all the
positions of the actual path. The directions which in the
medium suffer the maximum elongations (or contractions) will
rotate in the medium from one position of the actual path to
another, and there will be no coaxiality in the medium (nor
in space) between all the various tensors denoting the pure de-
formations, and their time derivatives (deformation velocities,
accelerations, etc.), each of these tensors having its own
orientation in the medium (and in space), and this orientation
is in general different from that of any other of the tensors.

(Illustrations by simple two-dimensional shearing movements of finite amounts)

The distinction between the privileged path and the actual one is not apparent when only two positions in the movement of a medium are considered (as is done in most textbooks), nor does it play a role in the classical theories of infinitesimally small deformations. It is, however, of paramount importance in the theories of deformations of all sizes (large or small), and in particular in the linking up of the movements and the forces coexisting in a differential cell of the medium.

2. Special Movements

In the movements of a medium through an ordered series of positions (0) (1)...(C)...(n)...(l̄), there are two special cases of particular interest, one in which the successive movements (C(C+1)) are all purely rigid, and the other when they are all purely deformatory.

For the case in which all the successive movements (C(C+1)) are purely rigid one has

$$\Psi = \Omega \quad \text{and} \tag{30}$$

$$\psi = \omega \quad \text{with} \tag{31}$$

$$I = \Theta_a = \Theta_p = \vartheta_a = \vartheta_p$$

so that in this case the superposition laws (22) to (24) simplify to

$$\bar{\Omega} = \Omega \cdot \omega \qquad \text{with} \qquad (32)$$

$$\bar{\Theta}_a = \bar{\Theta}_p = I \qquad (33)$$

This means that purely rigid movements in an ordered succession form a group since the resultant movement is purely rigid, but the group is, in general, not commutative ($\Omega \cdot \omega \neq \omega \cdot \Omega$) except when the rotations of the successive movements are coaxial (and $\Omega \cdot \omega = \omega \cdot \Omega$). Thus, in general, there will be no possibility of introducing here additive kinematic variables while in the exceptional case of coaxiality. Such variables can be introduced, and are found by taking the logarithms of the superposition law viz.

$$\ln \bar{\Omega} = \ln \Omega + \ln \omega \qquad \text{(for } \Omega \text{ coaxial to } \omega\text{)} \qquad (34)$$

It can be shown that the logarithm of the orthogonal tensors are antisymmetric tensors which represent the angles of rotation, and these are additive for successive coaxial rotations.

For the case in which all the successive movements ($C(C+1)$) are purely deformatory the result is of much greater complexity. One has here

$$\Psi = \Theta_a = \Theta_p = \Theta \qquad \text{and} \qquad (35)$$

$$\Psi = \vartheta_a = \vartheta_p = \vartheta \qquad \text{with} \qquad (36)$$

$$I = \Omega = \omega \qquad (37)$$

and this gives a superposition law

$$\bar{\Psi} = \Theta \cdot \vartheta \tag{38}$$

or resolved

$$\bar{\Omega} = \Theta \cdot \vartheta \cdot (\vartheta \cdot \Theta^2 \cdot \vartheta)^{-\frac{1}{2}} \tag{39}$$

$$\bar{\Theta}_a = (\Theta \cdot \vartheta^2 \cdot \Theta)^{\frac{1}{2}} \tag{40}$$

$$\Theta_p = (\vartheta \cdot \Theta^2 \cdot \vartheta)^{\frac{1}{2}} = \bar{\Omega}^{-1} \cdot \bar{\Theta}_a \cdot \bar{\Omega} \tag{41}$$

This means that purely deformatory movements in an ordered succession do not form a group, and do not commute as $\bar{\Omega} \neq I$, and $\Theta \cdot \vartheta \neq \vartheta \cdot \Theta$, except when the successive purely deformatory movements Θ and ϑ are coaxial when one finds $\bar{\Omega} = I$ and $\Theta \cdot \vartheta = \vartheta \cdot \Theta$. Again, in general, it will not be possible to introduce here additive kinematic variables, but such variables can be introduced for the exceptional case of coaxiality by taking the logarithm of the supermotion law, viz.

$$\ell n \, \bar{\Theta} = \ell n \, \Theta + \ell n \, \vartheta \tag{42}$$

(for coaxiality of Θ and ϑ)

D. The Kinematic Variables

The kinematic variables here introduced may serve for a description of the movements in terms of purely rigid and purely deformatory components, as given by the vectors of translation and the various tensors of rotation and deformation. It will be completely arbitrary, and a matter of convenience, whether one uses the said vectors and tensors, or any scale functions of these. In fact, various investigators have used different such scale functions, including the logarithms of the tensors Θ_a and Θ_p , or the squares of these tensors, etc., or the squares minus unity, etc.

The independence of the arbitrary choice of the scale function will have to be considered for the establishment of the relations between the movements and the forces, and it will be seen that these relations will appear in the simplest form when some particular scale functions are chosen, and the movements and forces are referred to the same geometric elements.

E. Differential and Integral Calculus of Movements

The differential and integral calculus in its simplest form applies to variables whose increments are additive, and hence commutative. When the calculus is used for the description of movement with rotations and deformations of any size, large or small, complications arise because the kinematic variables used in this description have increments which, in general, obey nonadditive and noncommutative laws of superposition and resolution, and these have to be taken into account in the application of the calculus. A typical example of the complications which arise here can be seen in the definition of the deformation velocity by time differentiation, and in the calculation of the total deformation by a time integral. In the time differentiation one has to distinguish between the differentiations of Θ_a , Θ_p , ϑ_a and ϑ_p , each of which defines some deformation velocity, and neither of these gives by integration over the time the value of the total deformation $\bar{\Theta}_a$ or $\bar{\Theta}_p$ which

the medium has suffered. The complications mentioned above do not arise in the special movements, discussed in the previous sections, when these movements are described in terms of the kinematic variables with commutative and additive increments according to (34) and (42).

Seminar on Continuum Mechanics

Lecture V

The Principles of Dynamics

A. Introduction

   1. Different Approaches to the Problem of the Mechanical
       Forces

B. Choice of Systems of Reference

   1. Galilei Systems

   2. Non-Galilei Systems

C. The Principle of Equilibrium of Forces

   1. Equilibria in Rigid Mechanics

   2. Equilibria in Continuum Mechanics of Deformable Media

       (a) The Definition of the Traction Forces and Stresses
           in Relation to Test Areas

       (b) Equilibria of Traction Forces Acting on Test Areas
           through a Point

       (c) Equilibrium of All Forces Acting on the Mass and
           the Surface Areas of a Differential Cell

D. The Dynamical Variables

   1. Goniometry of Dynamics

   2. The Superposition Law

E. Application to Boundary Problems

Seminar on Continuum Mechanics

(Lecture V)

Principles of Dynamics

A. Introduction

   1. Different Approaches to the Problem of the Mechanical

       Forces

       Dynamics is concerned with the distributions in space

and time of the mechanical forces which will be described in

terms of dynamical variables.

       In spite of several attempts at unification, it has

not yet been possible to present all the different kinds of

mechanical forces under a single aspect.

       The most promising approach in this direction has

been made by Einstien in his theory of general relativity in

which he introduces a certain space-time metric (i.e., a certain

set of yardsticks, clocks, and rules of measurements) which

allows to account for the mechanics of rigid media in terms of

this metric only without introducing forces of any kind. Un-

fortunately, the theory is limited to rigid media, and there

has been no success in trying to deal with the general continuum

mechanics of nonrigid (i.e., deformable) media in terms of geome-

try only, i.e., without introducing forces. For this reason

we shall exclude a discussion of Einstein's general relativity

theory from the Seminar on Continuum Mechanics.

An entirely different approach has been made by La-
grange. He did not try to avoid the introduction of forces but,
on the contrary, developed a mathematical formalism by which it
is possible to replace any geometric conditions imposed on the
freedom of movement by an equivalent set of forces, referred
to as the Lagrange forces of restraint. Some examples of these
forces have already been mentioned in the lectures on the Kine-
matics where it was shown that any invariants in a movement
reduce the freedom of movements, and this reduction is equiva-
lent to the introduction of infinitely strong Lagrange forces
of restraint which "keep the invariants invariant," i.e., pre-
vent their changes in any way. (See the Trellis Model.) We
shall use Lagrange's concept of forces of restraint throughout
the lectures in this Seminar, as this concept has proved itself
to be very useful for the formulation of the laws of Continuum
Mechanics. The Lagrange forces of restraint defined so far
have been limited to forces which do not produce any work as
they extend in directions along which the movement is zero or
in directions which are in an algebraic sense perpendicular to
the movement. It seems possible, although it has not yet been
done, to proceed further and define all the other mechanical
forces, i.e., the mass forces of inertia and gravity and the
traction forces of elasticity and viscosity as Lagrange forces
of restraint irrespective of whether they do or do not produce

work. This could be done by introducing certain systems of reference arbitrarily as privilege systems (they are termed Galilei systems of reference), and define relative to these systems first the Newtonian forces of inertia, as forces of restraint, and then define as mechanical forces of restraint all quantities which can equilibrate the inertia forces. Even if one adopted this procedure the whole scheme would still involve the arbitrariness of the introduction of the Galilei systems of reference, and it would also lack any principle which would allow us to determine how many kinds of mechanical forces have to be considered and what kinds one should choose. We shall therefore not use in the lectures of this Seminar such a generalization of the Lagrange forces of restraint but introduce the mechanical forces according to a third approach to the problem.

The third approach here considered is based on Newton's three laws of inertia, on the principle of equilibrium of forces, and on the Galilei systems relative to which these laws are valid. It will be assumed that Newton's laws, as well as the derivation and formulation of the principle of equilibrium of forces, and the Galilei systems of reference are well known from the textbooks so that it will suffice here to refer to some specific items where difficulties are often encountered. This will be done in the following sections.

B. Choice of Systems of Reference

    1. Galilei Systems of Reference

        With respect to Newton's laws of inertia, the various possible systems of reference are not equivalent, and there is one privileged group, known as the Galilei systems of reference whose measuring rules and devices ("rigid" yardsticks and mechanical clocks) are calibrated in accordance with Newton's laws of inertia.

        The Galilei systems of reference can be transformed into one another by a group of transformations known as the Galilei transformations. They are linear in the coordinates of space and time, and are found as the linear transformations which leave Newton's laws of inertia invariant.

        The whole of the Mechanics is then so formulated relative to any chosen Galilei system that all the laws of Mechanics remain then invariant under the group of Galilei transformations. Thus, the Galilei group of transformation defines the space and time symmetries in Mechanics which are of fundamental importance in all mechanical problems.

        The symmetry in space only consists of three different types of symmetry operations, _viz._, translations, notations and mirror inversions, and of all combinations of these operations. The symmetries corresponding separately to the three types of operations are referred to as Homogeneity, Isotropy, and Parity.

The space time symmetry consists of one type of operation only, <u>viz</u>.,

$$x' = x - ct$$

$$t' = t - \frac{x}{c}$$

and of combinations and repetitions of this operation in all three dimensions in space, when c represents a constant velocity, t an arbitrarily chosen fixed instant of time, while x and t, and x' and t' represent the coordinates in space and time before and after the transformation.

2. Non-Galilei Systems

When a system of reference is chosen which is fixed to the earth, or to any material body which moves relative to a Galilei system in a manner different from that allowed by the Galilei transformations, then it can be transformed from a Galilei system by a transformation which lies outside the group of Galilei systems, and thus a non-Galilei system is produced.

The working with such systems is rather difficult because one has here to introduce new fictional additive forces before one can apply the equilibrium of forces or other laws of mechanics. These fictitious forces are often referred to as Corriolis forces.

As an example consider the movement of a pendulum (Foucauld's pendulum) relative to a system of reference fixed

to the earth. Because of the rotation of the earth the plane

of movement of the pendulum does not appear to be invariant

but to rotate relative to the chosen (non-Galilei) system of

reference, and thus to trace a sort of Lissajou's figure on a

horizontal plane.

C. The Principle of Equilibrium of Forces

Once the Newtonian forces of inertia have been intro-

duced relative to a Galilei system of reference, one can pro-

ceed to a definition of mechanical forces of different kinds

by the assumption of a principle of equilibrium of forces, ac-

cording to which there is an equilibrium of all mechanical

forces, including the forces of inertia under all conditions

of rest or motion, and at every point in the medium for all

surfaces through the point, and all portions of the medium.

By this principle of equilibrium one can define any physical

quantity as a mechanical force if it can equilibrate the inertia

forces, and one can measure the mechanical force by its equal

and opposite inertia force. In this introduction of mechanical

forces one has to rely on empirical data to determine the number

and kind of mechanical forces which are required for the formu-

lation of the laws of Continuum Mechanics.

1. Equilibria in Rigid Mechanics

In the theory of rigid mechanics one considers in addi-

tion to externally applied driving forces only the mass forces,

i.e., forces which act on the mass of the rigid bodies.

From the empirical knowledge of free falling objects on the earth, and of the movements of the planets round the sun one defines and measures the forces of gravity by their equilibrium with the forces of inertia. There is then also an equilibrium of the moments of forces round any arbitrarily selected axis of rotation.

In considering these equilibria it is important to note that the inertia forces are proportional to the acceleration, i.e., to the second time derivative of the displacement, irrespective of the value of the displacement and of other time derivatives. In particular, the inertia forces can be present at an instant when the velocity is zero, i.e., when there is no movement, such as for a pendulum at the instant when it changes from an upward to a downward movement and has at that instant the velocity zero. At that instant the inertia force of the pendulum has even its maximum value, while it is zero when the pendulum passes through its equilibrium position, and its velocity has its maximum value.

   2.   Equilibria in Continuum Mechanics of Deformable (Nonrapid)

            Media

        (a) The Definition of the Traction Forces and Stresses

              in Relation to Test Areas

        In the Continuum Mechanics of deformable media the empirical knowledge of the behavior of such media has made it

necessary to introduce in addition to the types of mechanical

forces already considered in rigid mechanics a new type of

mechanical forces, called traction forces, which act at every

point in the medium across a surface area passing through the

point.

To define and measure the traction forces exactly,

one has to consider at every point in the medium the conditions

under which they can equilibrate the inertia forces (since all

mechanical forces are defined and measured through such equi-

libria). To produce these conditions one uses an idealized

imaginary process by which an infinite small sphere of the

medium surrounding the point considered is bisected by a plane

into two equal parts. According to Newton's first law of in-

ertia there will be equal and opposite traction forces acting

on the areas of the two sides of the bisecting plane, and these

traction forces would drive the two parts of the medium away

from one another, or press them against each other. The strength

and direction of the traction forces can then be measured by

applying exterior forces to the two parts of the sphere, ob-

serving the resultant accelerated movements of the two parts,

and calculating the traction forces as the differences between

the externally applied forces and the inertia forces. (In the

simplest case the external forces are so chosen that no accelera-

ted movement of the two parts results, and there the inertia forces

are zero and the traction forces are measured by the compensating

external forces.) One finds that the traction forces are pro-
portional to the area on which they are tested, if the size of
the test area is infinitesimally small, approaching to zero
in the limit. At every point in the medium one then considers
the traction forces across test areas of all orientations
round the full solid angle of directions and defines at every
point in the medium the stress by the operator (tensor) which
acts upon the test area and transforms it into the traction
force across that area, _viz._,

$$dF^m = dB^{ij} Q_{ij}^{\cdot\cdot m} \tag{1}$$

where one denoted by

dF^m the components of the infinitesimally small traction

        force, defined as contravariant vector[1]

dB^{ij} the components of the infinitesimally small area tensor

        defined as double contravariant antisymmetric

---

1 All forces are usually defined as covariant vectors. This
definition arises from the postulate that the energy should be
scalar invariant, resulting from the scalar multiplication of
the force with the contravariant vector which conventionally
represents the path along which the force acts. For our pur-
poses it will be more convenient to define the traction forces
as contravariant vector by postulating its equilibrium with
the inertia force which as mass times acceleration is repre-
sented by a contravariant vector.

tensor of second order, and by

$Q_{ij}^{\cdot\cdot\cdot m}$ the mixed component of the stress defined as a tensor of third order, which again is antisymmetric in the indices ij (since any symmetrical part in these indices can be equated to zero as its contribution to the result of the multiplication (1) is zero).

One can simplify equation (1) with the help of the two tensors of third order which represent the element of volume and of reciprocal volume (see, e.g., Brillouin l.c.). By multiplying with one or other of these tensors and contracting over the indices ij one obtains pseudotensors which will be indicated by a stroke at upper index level for volume densities and at lower index level for volume capacities of tensors. For the area one obtains a volume capacity of a tensor of first order (often referred to as an "axial vector") with covariant components dA_{k-}, say, and for the stress a volume density of a symmetric tensor of second order in double contravariant components P^{km-}, say, viz.,

$$g^{-1/2} \, dA_{k-} = dB^{ij} \quad \text{and} \quad (2)$$

$$g^{-1/2} \, P^{km-} = Q_{ij}^{\cdot\cdot\cdot m} - Q_{ji}^{\cdot\cdot\cdot m} \quad (3)$$

where $g^{-1/2}$ and $g^{1/2}$ denote respectively the reciprocal volume and the volume of the unit cell in a coordinate system with a metrical tensor g_{ik}. (The multiplication of the tensor capacities and tensor densities with these factors insures invariance of the equations against all transformations of the

coordinate system). It will be understood the indices i,j,k represent in a righthanded system cyclical permutations of the indices 1,2,3, and in a lefthanded system the indices of cyclical permutation of 3,2,1, with the normal to the test area being counted positive in the outward direction.

In all the problems of continuum mechanics it has not been necessary to introduce any further forces, (such as may act on a line through the point of the medium considered, and may be proportional to the length of that line) so that in the equilibria encountered in the mechanics of deformable media, one has to consider only the mass forces (of gravity and inertia) and the external driving forces which are already known from the mechanics of rigid media, and the newly introduced traction forces, as defined above.[1]

When one applies the principle of equilibrium of forces one has to distinguish between two cases, according to whether or not the traction forces considered refer to surface areas which pass through the same point, or surface areas which enclose a mass of the medium such as the surface areas of a differential cell. In the former case the traction forces must

---

1. Problems in which one has to consider surface tension effects are regarded here as outside the scope of the continuum mechanics as discussed in this series of lectures. To account for surface tension effects one has to introduce additional forces corresponding to that tension.

establish equilibria among themselves since the mass forces
are then zero because the masses involved are zero.[1] In the
latter case the equilibrium will be established between the
traction forces across the surface areas of the differential
cell and the mass forces of inertia and gravity acting on the
mass of the cell.

(b) Equilibria of Traction Forces Acting on Test Areas

Through a Point

At every point in the medium one has to consider three
equilibria of the traction forces which are conveniently written
in terms of the stresses, i.e., of the traction forces reduced
per unit area, and considered at every point on bisecting planes

---

1. One can approximate to the conditions of this case by taking
a differential cell and going to the limit of the dimensions
of the cell approximating to zero. The traction forces then
vanish with the area, i.e., the second power of the cell edges
while the mass forces vanish with the volume, i.e., the third
power, and can therefore be here neglected against the traction
forces. This procedure, which is adopted in all the textbooks
is less satisfactory than the one suggested above, since the
textbook procedure does not make it clear when the mass forces
can be neglected, and when they have to be taken into account
in a differential cell.

of all orientations round the full solid angle of directions.

The first equilibrium results from the definition of
the traction forces and stresses acting across the two sides
of the test area of the bisecting plane, viz.,

$$p^{mk-} (dA_{k-} + dA'_{k-}) = 0 \text{ for } \begin{array}{l} g^{-1/2} dA_{k-} = dB^{ij} \\ \\ g^{-1/2} dA'_k = dB^{ji} \end{array} \tag{5}$$

A second equilibrium results from the assumption of a
continuity principle for the traction forces. This principle
postulates (similarly to the one established for the kinematics)
that the traction forces should be analytical functions of the
area vectors and hence developable in the neighborhood of each
point in the medium into a convergent power series with constant
coefficients. Neglecting the higher powers against the linear
terms in the limit of all area vectors emanating from the same
point, one finds equations (1) to (3). The equilibrium deduced
from these equations is usually referred to as the equilibrium
on a tetrahedron. It would, however, be more accurate to speak
of the equilibrium on four planes passing through the point,
parallel to the faces of a tetrahedron and having test areas
equal to the sizes of the said faces. Taking account of the
conventional assignments of positive and negative values to these
test areas, one then finds

$$p^{mk-} (dA_{k-} + dA'_{k-} + dA''_{k-} + dA'''_{k-}) = 0 \text{ for}$$

$$(dA + dA' + dA'' + dA''') = 0 \tag{6}$$

There is a third equilibrium to be considered which is usually referred to as the condition of symmetry of the stress, and which may be formulated as

$$dA_{m-} \; P^{mk-} - dA_{k-} P^{km-} = 0 \tag{7}$$

The three equilibria described above define conditions of the traction forces and stresses at any point in the medium across planes of all orientations passing through the point in all directions round the full solid angle.

(c) Equilibrium of All Forces Acting on the Mass and Surface Areas of a Differential Cell

The traction forces (and stresses) have been considered in the previous paragraph in their variation at a point round the full solid angle across planes which all pass through the same point. Now, it will be necessary to consider the variation of these forces from one point to another along the edges of a differential cell. It is found that the traction forces acting on the three pairs of parallel faces of the cell have differences which are proportional to the lengths of the edges of the cell, and that a reduction of these differences per unit length produces a force which is proportional to the volume, and hence to the mass of the cell. An equilibrium is then established between this mass force, and the mass forces of gravity and inertia, viz.,

$$\frac{\partial}{\partial x^m} |g|^{\frac{1}{2}} P^{mk-} + G^k + J^k = 0 \tag{8}$$

where G^k and J^k are the components of the contravariantly defined forces of gravity and inertia.

D. The Dynamical Variables

1. Goniometry of Dynamics

The dynamical variables are introduced in order to describe in a quantitative way the distributions in space and time of all the mechanical forces. This description is given in terms of the externally applied forces and all the mechanical forces which enter into the equilibria discussed in the preceding paragraphs. It follows from these equilibria that the goniometry of the forces again provides the key to such a description, as all the equilibria refer to the infinitesimally small neighborhood of a point in the medium in the limit of the dimensions of the differential cell approximating to zero.

It is conventionally agreed that at each point in the medium one chooses as dynamical variables the components of the tensors and pseudotensors of various orders which describe the mass forces of inertia and gravity, and the traction forces, or stresses present at the point. To describe then the distribution through space and time, one proceeds by passing from the tensors defined at a point to appropriately extended fields of these tensors.

2. The Superposition Law

If one superimposes two states of stress in the neighborhood of the same point in the medium, the resultant state of stress will be the sum of the two superimposed ones, if all the stresses are measured in terms of the tensor density P^{km-}. In short one finds here an additive superposition law, viz.,

$$P^{km-} + p^{km-} = \bar{P}^{km-}$$

where P^{km-} and p^{km-} denote the two superimposed states of stress and $\bar{P}^{km-}$ the resultant one. This makes it possible to apply to the dynamical variables the differential and integral calculus without any of the complications encountered in the application of the calculus to the kinematic variables (see preceding lecture).

The conventional choice of the dynamical variables P^{km-} is, of course, arbitrary and may be replaced by the choice of any scale function of P^{km-} whenever this appears to be advantageous for algebraic calculations or graphical representations. In practice, scale functions have been introduced only for graphical representation of the stresses (there are in classical theory four different quadrics used for such representations), while for algebraic purposes one uses the conventionally agreed pseudotensors, because they already obey a simple (additive) superposition law, and the introduction of scale

functions here would necessitate a more complicated formulation of this law.

E. Application to Boundary Problems

The dynamic conditions existing at the boundaries of the medium under consideration require a special study.

Two extreme cases may be mentioned here by way of examples, one referring to free boundaries, and the other to rigid boundaries. In both cases the conditions are assumed to be so chosen that the mass forces of gravity and inertia can be neglected, and that the sample of the medium is a plane sheet with a circumferential boundary which in some parts is free and in other parts fixed in rigid grips. One may then use the grips to apply to the gripped parts of the boundary, in the plane of the sheet forces and/or displacements of any direction and amount, and inquire into the states of stress at the free parts of the boundary, and at the parts held in the rigid grips.

By applying the principle of equilibrium of forces, it can then be shown that the state of stress at every point of the free part of the circumferential boundary is a simple pull (or push) tangential to the circumference at that point, irrespective of the directions and amounts of the forces and/or displacements applied to the gripped parts of the boundary, and irrespective of the form of the circumference and of the mechanical properties of the medium.

It can be similarly shown that the state of stress at every point of the rigidly held part of the boundary is such that the stress component normal to the boundary always approaches to zero, except in an area near the corners of the boundary. The size of that area will depend on the properties of the medium, and on the forces and displacements applied to the grips.

Seminar on Continuum Mechanics

Lecture VI

The Principles of Mechanical Properties in Continuum Mechanics

Part I. Instrumentation for the Measurement
of Mechanical Properties

A. Introduction

1. Theoretical Aspects

2. Empirical Aspects

3. The Goniometric Design of Instrumentation

B. Descriptions of Instruments

1. The Rheogoniometer

2. A Two-Dimensional Straining Device

The Principles of Mechanical Properties in Continuum Mechanics

Part I.   Instrumentation for the Measurement
of Mechanical Properties

A.   Introduction

1.   Theoretical Aspects

In Continuum Mechanics use is made of the transforma-
tion theory of metrical systems of reference, on the basis of
which certain principles of symmetry, of continuity and of equi-
librium of forces are derived for the Kinematics and Dynamics as
set out in the previous lectures. These principles are regarded
as fundamental, and provide the theoretical aspect for the in-
strumentation of the metrical system of reference, and for the
mechanical properties here conceived as the parameters which
interrelate the kinematic and dynamical variables that coexist
in the medium at every instant of time and every point in space.
The said theoretical aspect, however, provides for the instru-
ments only the laws of calibration, and an empty systematic
register for the infinite variety and complexity of mechanical
properties in accordance with all the possible linear and non-
linear interrelations between the coexisting kinematic and dynam-
ical variables. It will therefore be necessary to supplement
the theoretical aspect with an empirical one which would use as

a system of reference instrumentation which was designed and calibrated in accordance with the theory, and which would allow to fill the empty theoretical register for the mechanical properties with observed numerical data.

2. The Empirical Aspect

The problem of suitable instrumentation for the empirical aspect has been approached in two different ways. One approach starts with the consideration of the kinematic and dynamic conditions under which the medium is used in practice, and then proceeds to the development of instrumentation in which measurements can be taken of the kinematic and dynamic variables while the medium is subjected to experimentally controlled conditions which approximate as closely as possible to those under which the medium is to be used in practice.

This approach is widely used in industry for the control of production processes, for the working out of specifications, etc., and is there of great value. However, we shall not deal with it here since the results obtained in such measurements can contribute only little to a better understanding of the mechanical properties of the medium which are revealed in a more comprehensive manner by the second approach. Here one starts from the theoretical conception of mechanical properties as the parameters which specify the relations between the coexisting kinematic and dynamic variables and which can be determined as

coefficients of the development of the relations into conver-
gent power series. The instrumentation is then so designed as
to approach to an ideal which takes into account the distribu-
tion of the kinematic and dynamic variables in time and space
so as to cover for every point considered

(a) the whole solid angle of directions in space and
the development in time

(b) the whole range of kinematic and dynamical condi-
tions considered

(c) the whole range of different mechanical properties
of media which are compatible with the fundamental principles
of kinematic and dynamics.

3. The Goniometric Design of Instrumentation

For the design of instrumentation approaching to the
said ideal, it is important to note that one can describe
as fields of tensors of various orders all the kinematic and
dynamic variables, as well as the parameters which specify the
interrelations between these variables, and which determine the
mechanical properties. (This follows from the transformation
theory in combination with the principles of preservation of
continuity and equilibrium of forces, as set out in the lectures
on the kinematic and dynamics .)

The tensor character of all the quantities to be meas-
ured suggests that the Goniometry should be the keynote of the

design of the instrumentation for the measurement of the mechanical properties of a medium, just as the goniometry dominated the determination of the kinematic and dynamic variables. The advantages of such a design are that the measurements of the components of the said variables across three mutually perpendicular planes will suffice for a complete determination of the values of the components at the point considered round the full solid angle of directions, and for an equally complete determination of the mechanical properties there.

The conventionally designed instruments such as the capillary viscometers, Conette instruments, etc., can neither singly nor in combination provide the means for measurements to be taken across three mutually perpendicular planes, so that it had become necessary to design instruments of a goniometric type which are referred to as "Rheogoniometers."

B. Description of Instruments

1. The Rheogoniometer

The actual instruments constructed as Rheogoniometers approach to the ideal but are still far from achieving it.[1]

1. The first model was designed by the author in collaboration with Mr. S. M. Freeman, while later improved models were developed in collaboration with Mr. J. Roberts, the Ministry of Supply and the Manufacturers, Messrs. Farol Research Engineers, North Bersted Bosnor-Regis, Sussex, England.

The instrument, diagrammatically illustrated in Fig. 1, consists of a vertical, high precision vee-bed carrying the slide on which the torsion measuring head 'P' is mounted. The axis of the torque and the gap for the material under test are thus located to the required high degree of accuracy.

The material under test is held in a gap 'A' between two boundary members enclosed in a thermostatically controlled chamber 'F'. The lower member is connected to the drive, and allowed to rotate in the bearings 'B' and 'B1' of the bottom mounting 'M'. The vertical vee-bed is statically clamped to the bottom mounting 'M' and the torsion measuring head 'P' can be moved up and down the axis by means of a precision lead-screw for an exact control of the gap size. The two members have preferentially the form of a flat plate and a nearly flat cone which ensures for all points throughout the conical gap a rate of shear constant to within plus or minus two per cent. Various other forms can be provided for the two members, such as a pair of flat plates, or concentric cylinders, or cones with the same or slightly different cone angles. For most experiments it is convenient to use members made of light alloy, but for certain special purposes (see below) it is necessary to use as the upper member a glass head 'R' with a row of capillary gauges along one diameter.

The lower cone can be given one of four types of motion:

(a) a unidirectional rotation at a steady rate:

(b) an oscillatory harmonic motion of predetermined amplitude and frequency.

(c) a motion by superimposing (b) on (a)

(d) a predetermined acceleration or deceleration to any of the above states of motion.

(a) The unidirectional rotation is provided by one half h.p. synchronous motor through gearboxes and an electromagnetic clutch. The gearboxes are designed to give speed ratios of 1:1 down to 1,000,000: 1 in sixty geometric steps. The drive from these gearboxes is transmitted through the electromagnetic clutch which may be operated quickly by discharging condensers into it, or at a slower rate by switching in preset resistors. The gearboxes give a range of platen speeds from .00454 revolutions per hour to 60 revolutions per minute.

(b) The oscillatory motion is provided by a second motor and gearboxes via flexible couplings to eliminate vibration transmission and operates a variable throw mechanism to give a frequency range from .1137 cycles per hour to 1500 cycles per minute. The amplitude of oscillation may range from 0 to .050 inches, that is from 0 to .03 radians on the lower cone.
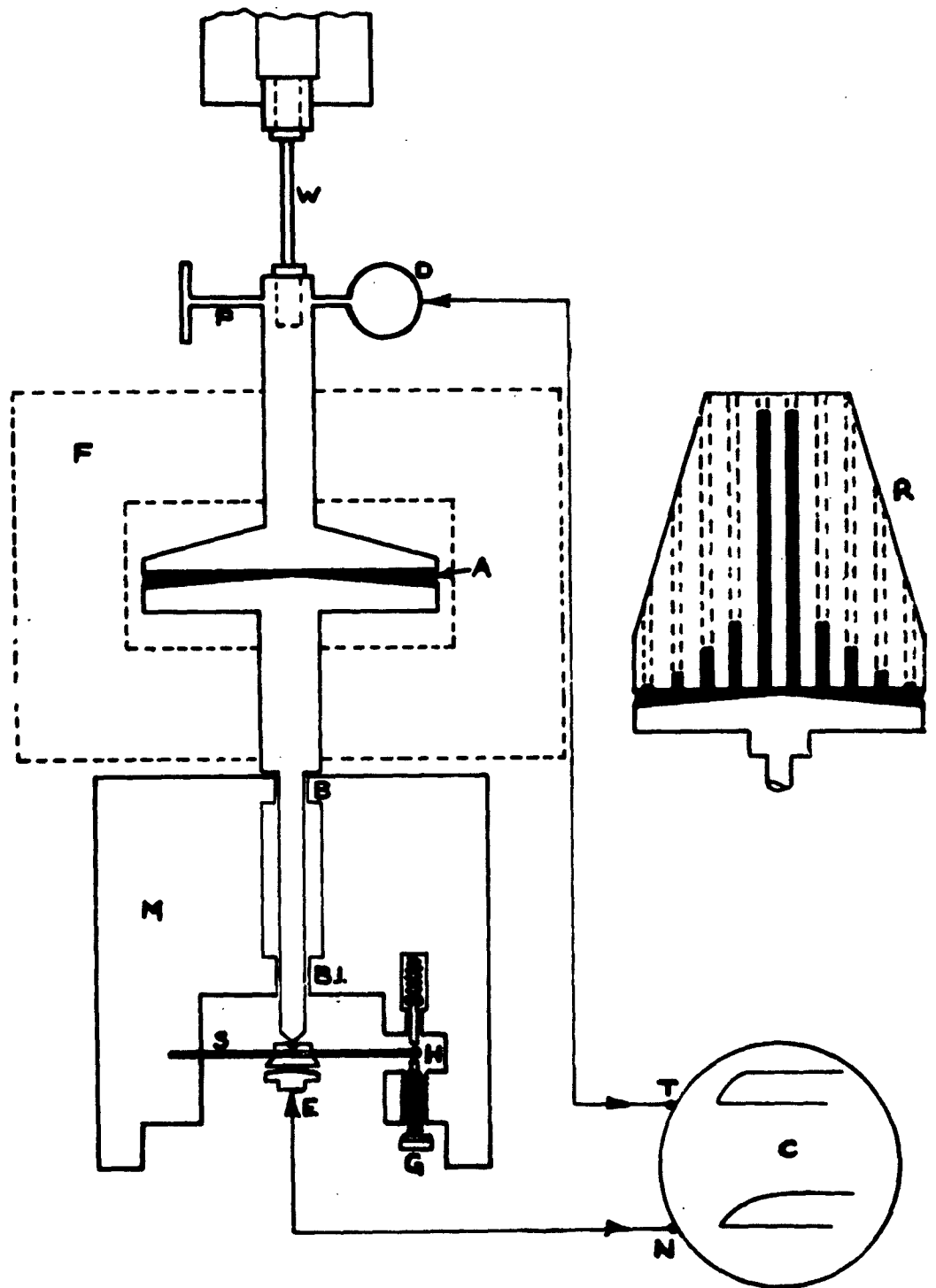
Fig. 1

It has been arranged that this adjustment does not alter the phase of the motion.

(c) Both drives are combined to give type (c) motion, as the final coupling to the journal is by worm and wheel. The rotational motion is imparted by turning the worm whilst the oscillatory motion results from an axial movement of the worm.

The latest model was designed for the specific purpose of combining ease of handling with the required high degree of accuracy over the wide range of conditions likely to be encoun- tered in practical investigations. The model was intended to be used mainly for mechanical actions involving a torsional shearing movement, but supplementary equipment can be made to experiment with other actions such as simple pulls, unidirec- tional elongations, etc.

Measuring Devices. The measuring devices have been designed for a complete determination in space and time of the movements and forces in the material. The movement of the mate- rial in the gap can be determined from that of the moving member by a linear interpolation between this member and a static one. This linear interpolation has been shown to be correct in a flat gap of conical shape 'A' and a narrow width to about plus or minus one per cent. In both rotational and oscillatory movements the speeds or frequency can be quickly ascertained from the

selected gear position, thus providing a time base for the recorded movement; in addition, for the oscillatory movement, an indication of input frequency and amplitude can be obtained from one end of the wormshaft. By using the glass head 'R' one is able to make a direct check of the movement of the material in all three dimensions of space by observing through a microscope the movements of small particles of dust or small air bubbles suspended in the material. For the determination of the forces, electrical transducers 'D' and 'E' are used to measure on the upper member the torque against the torsion wire 'W', and on the lower member the thrust against the spring 'S'. The measurements are amplified by Boulton Paul Multimeter instruments and displayed on the screen 'C' of the cathode ray oscillograph or on the chart of a pen recorder. Typical traces are shown in 'T' and 'N'. In order to cover a wide range of forces, the torsion wire 'W' and the spring 'S' can be replaced by others of different stiffness coefficients. The spring is supplied with a sensitivity control 'GH'. In most cases the measurements of torque and thrust suffice to calculate in the conical gap the forces in the material at all points, and across planes of all orientation in space.

The calculations have been verified experimentally for a large variety of materials and conditions, but remain open to doubt for any new material and set of conditions not previously

tested. To remove any doubt, additional experiments must be made. These are two types and are discussed hereunder for liquid materials which under shear are stable for a sufficiently long time to allow pressure measurements to be made in the slow recording capillary gauges. The experiments of the first type use, as the upper member, the glass measuring head 'R' whose capillaries along a diameter indicate the distribution of normal pressures across the shearing plane of the torque as illustrated in the diagram 'R'. The experiments of the second type use a surround attached to the lower member, and a standardized Newtonian liquid (such as water, etc.). This liquid is kept in contact with the material under test along a cylindrical interface formed in the gap at various radial distances, so that for each distance the hydrostatic pressure in the liquid measures the pressure of the material normal to the interface. By combining all the experimental results, one is able to determine the complete distribution of forces in the material from first principles without any supplementary assumptions.

The instrument is suited to the testing of the mechanical properties of materials of various types, including rayon, plastics, lubricating oils, greases, natural and synthetic rubbers, printing inks, paints, adhesives, gelatine, soaps, cosmetics, clays, waterglass, dairy products, dough, honey, etc.

Testing Methods. The testing methods are mainly of torsional shear. They can be varied so as to include static conditions, steady motions at various rates of shear, and torsional vibrations of various frequencies and amplitudes. All measurements can be read off on direct reading devices so that tests can be carried out speedily and with great ease.

For routine tests (such as occur for instance in quality checks in production control) it suffices to carry out only a few standardized experiments. While the usual instruments measure in each such experiment only one property, the apparent viscosity, the Rheogoniometer measures two properties, namely, the apparent viscosity and apparent elasticity (rigidity), thus obtaining twice the information usually available.

For tests in research and development it is possible to multiply manifold the information usually available. Approach can even be made to a complete characterization of the materials in respect of all its shear properties. A first approximation -- giving the most significant shear properties -- can be achieved by making vibrational experiments over the whole range of available frequencies and amplitudes. A second approximation can then be arrived at by making supplementary experiments under static conditions and measuring the relaxation of stress in the material. The use of vibrational experiments as the basic method of testing is strongly recommended, not only because it

approximates to a complete characterization, but also because it has the following further advantages:

(i) It causes a minimum of disturbance to the colloidal and chemical structure of the materials in its initial state, and thus gives the nearest approach to a characterization of that structure in terms of mechanical properties. It is even possible to characterize in this way structures which are highly thixotropic and unstable under steady shearing movements (such as clay suspensions in water) because they remain stable in torsional vibrations over a wide range of frequencies provided the amplitude of strain is kept sufficiently small.

(ii) It allows an exact thermodynamical interpretation of the results as the work in vibration can be divided clearly into two parts, one completely reversible and elastic, and the other completely irreversible and viscous.

(iii) It provides a common base for the comparison of structures of different types and consistencies (solid, liquid and intermediate) as it is equally applicable to all of them.

A new development incorporated in the latest model enables an independent vibrational analysis to be made on a material in a state of steady share. This is particularly useful for the testing of strongly dilatant materials.

The new electro-magnetic clutch coupled with a 'hard' type of measurement enables accurate tests to be made on materials which are particularly temperature-sensitive, since the duration of an experiment may be as little as 0.1 of a second. This method is particularly suited to the testing of some printing inks with which, even in narrow gaps, it is difficult to dissipate the viscous heat at the required rate of shear.

2. A Two-Dimensional Straining Device

An instrument was designed and manufactured which can impose arbitrarily predetermined elongations or contractions along two mutually perpendicular directions. In particular one can produce homogeneous harmonically varying elongations and contractions with independently chosen amplitudes, frequencies and phases in the two mutually perpendicular directions. The instrument uses an extensible membrane, preferably a rubber sheet, as a base on which a specimen of the material under test is spread in the form of a thin layer. The specimen adheres to the surface of the rubber sheet which is extended (and/or contracted) in two mutually perpendicular directions by sets of parallel strips fastened to driving carriages which are moved in predetermined ways.

The instrument is suited to the study of fatigue in thin layers of materials, over very wide ranges of different fatiguing actions. Fatigue patterns develop showing cracks along preferential directions.

Seminar on Continuum Mechanics

Lecture VII

Principles of Mechanical Properties of Continuous Media

Part II.   Thermodynamics of Energy Transformations
with a Scalar Equation of State

A. Introduction

1. Various Concepts and Definitions of the Mechanical
Properties in an Equation of State

a. True Properties as Constant Coefficients in Con-
vergent Power Series

b. Apparent or Pseudo-Properties Dependent on Kine-
matical and/or Dynamical Variables

2. True and Pseudo-Properties as Operators on Either
Kinematic or Dynamic Variables

B. Thermodynamics of Energy Transformations

1. Transformation of the Energy of External Work into
the Free and Dissipated Forms of Energy

2. Adiabatic Transformations

C. The Derivation of a Scalar Equation of State

1. Introduction of Simplifying Assumptions

2. The Coexisting Kinematic and Dynamic Variables De-
fined as Coaxial Cartesian Deviators with
Additive Laws of Superposition

3. Formulation of Scalar Equation of State for Mechanic-
ally Determinate Media

4. The Expansion of the Scalar Equation of State into a
Cycle of Theories

Seminar on Continuum Mechanics

(Lecture VII)

Principles of Mechanical Properties of Continuous Media

Part II.   Thermodynamics of Energy Transformations
with a Scalar Equation of State

A.   Introduction

1.   Various Concepts and Definitions of the Mechanical

Properties in an Equation of State

a.   True Properties as Constant Coefficients in Convergent

Power Series

When a continuous medium is subjected to a mechanical

action, one finds at every instant of time and point in the

medium, a coexistence of the kinematic and dynamic variables,

and one can conceive the mechanical properties of the medium

as the parameters which specify the relation between the two

kinds of coexisting variables.   This relation is referred to

as the "Equation of Mechanical State," or "Equation of State"

for short, and is assumed to be developable into a convergent

power series.   One can then define the mechanical properties

of the medium in a quantitative manner by identifying them

with the coefficients in the said power series.   These coeffi-

cients are constants throughout the region of the convergence of

the power series, and thus characterize throughout this region

the true mechanical properties of the medium independently of

the applied mechanical action, and of the values of the coexist-
ing kinematic and dynamic variables.

   b.   Apparent or Pseudo-Properties Dependent on Kinemat-

        ical and/or Dynamical Variables

        When the true mechanical properties of a medium are

conceived and defined as above, one has the advantage of char-

acterizing the medium by constants, but this advantage has to

be paid for dearly by the great complexity of the characteri-

zation, involving as it does, an infinitely large number of

such constants. For this reason it has been found more con-

venient to specify the equation of state and characterize the

mechanical properties of the medium in a simplified manner by

a small (finite) number of so-called "apparent" or "pseudo-

properties" which are not constants but functions of the applied

mechanical action, and/or of the coexisting kinematic and dy-

namic variables.

        There have been two different approaches to such a

simplification, according to whether the development into a con-

vergent power series of the equation of state was reduced to a

pseudo-linear or a pseudo-quadratic form.

        According to the first approach one incorporates in

the linear terms all the terms of second and higher degree in

the power series. This can be done by replacing the constant

coefficients of the linear terms by power series of the varia-
bles. The resultant equation will then be pseudo-linear, and
the new coefficients of the linear terms will then be defined
as the apparent or pseudo-properties of the medium. A typical
example of this kind is the so-called "apparent" viscosity of
a non-Newtonian fluid.

According to the second approach one reduces the
power series by considering the tensor character of the kine-
matic and dynamic variables and by applying the Cayley Hamilton
equation which allows to express all powers of third and higher
degrees of a tensor by powers of lower degrees, and the three
scalar invariants of the tensor. The reduced equation will then
involve the kinematic and dynamic variables only in the powers
of degrees 0, 1, and 2, and will thus become pseudo-quadratic,
with coefficients which are functions of the three scalar in-
variants of the kinematic and dynamic variables, and which are
then defined as the apparent or pseudo-properties of the medium.
A typical example is provided by Markus Reiner's formulation of
general laws of elasticity and viscosity.

2. True and Pseudo-Properties as Operators on either Kine-
matic or Dynamic Variables

Whichever definition of the true mechanical properties
or pseudo-properties one may adopt, it will be best to conceive
them as operators, as has already been done, for the kinematic

and dynamical variables (see previous lectures). It is then
important to note that this conception makes it clear that one
should clearly distinguish between two distinctly different,
but equally admissible definitions of the mechanical properties,
according to whether they are defined as operators acting on
the various powers of the kinematic variables and so producing
the dynamical ones (by way of the full or reduced power series),
or as operators acting on the various powers of the dynamical
variables and producing the kinematical ones (by way of the
full or reduced power series of the inverted function). There
is a discrepancy between the two definitions given above be-
cause the coefficients of the power-development of the inverted
functions are not, in general, equal to the reciprocal values of
the power series of the original function. In the literature it
is often not clear which of the two definitions is meant, and
without such a clarification no physical meaning can be attached
to the mechanical properties as one would not know on what they
operate. A typical example is provided by vibrational testing
when pseudo-properties of elasticity and viscosity are derived
from the frequency-dependent complex modulus $\Gamma =$

$\Gamma_A (\cos \Gamma_\phi + i \sin \Gamma_\phi)$ where Γ_A and Γ_ϕ denote the ampli-
tude and phase of Γ respectively. When Γ is conceived as
an operator on the kinematic variable (deformation) to give the
dynamical variable (stress) one finds the modulus of elasticity

and viscosity given by ($\Gamma_A \cos \Gamma_\phi$) and $1/\omega$ ($\Gamma_A \sin \Gamma_\phi$)

respectively where ω denotes the frequency, while these moduli

are given by ($\Gamma_A \dfrac{1}{\cos \Gamma_\phi}$) and $1/\omega$ ($\Gamma_A \dfrac{1}{\sin \Gamma_\phi}$)

when the inverted complex modulus is used as operator on the

dynamical variable (stress) to produce the kinematical one

(deformation).

B. Thermodynamics of Energy Transformations

    1. Transformation of the Energy of External Work into the

            Free and Dissipated Forms of Energy

        The principles of macroscopic thermodynamics can be

used to clarify the energy transformations which occur in a

continuous medium where it is subjected to some mechanical ac-

tions. If and when these energy transformations can be ex-

pressed in terms of the kinematic and dynamic variables which

coexist in every differentiated cell of the medium, then one

can relate these variables to one another by way of the thermo-

dynamic principles, and so establish for the medium an Equation

of State formulated in the scalars which describe the energy

transformations in a differential element of space and time.

        When one applies mechanical actions to a given medium,

some external work will be expanded on every differential cell

and the energy corresponding to that work will be transformed

into other forms of energy. One then deduces from the principles

of macroscopic thermodynamics that in these energy transforma-
tions, one has to distinguish two forms of energy, one called
"free energy," which is completely reversible into external
work under a suitably chosen cyclical process, and a second
called "dissipated energy," which is completely irreversible
so that none of it can be retransformed into external work in
a cyclical process.

2. Adiabatic Transformations

When the mechanical actions are applied to the medium
under adiabatic conditions, the total energy of the system re-
mains constant so that in adiabatic transformations the time
derivatives of all the forms of energy considered must add up
to zero, viz.

$$- \dot{W} + \dot{\phi} + \dot{G} = 0 \tag{1}$$

where $\dot{W}$, $\dot{\phi}$ and $\dot{G}$ denote the work per second of the exter-
nally applied energy, and of the free and dissipated forms of
energy respectively. By definition $\dot{\phi}$ must be a total differ-
ential,[1] while $\dot{G}$ is not, and remains always positive.[2] This

1. This insures that the work of the free energy vanishes over
a closed cyclical process.
2. This insures that no part of the dissipated energy can be
retransformed into external work.

will be indicated formally by

$$\dot{\phi} = \frac{d\phi}{dt} \qquad \text{and} \qquad (2)$$

$$\dot{G} \geq 0 \qquad (3)$$

C. The Derivation of a Scalar Equation of State

1. Introduction of Simplifying Assumptions

For the derivation of a scalar equation of state, one

has to distinguish between two categories of media, referred to

as "mechanically determinate" and "mechanically undeterminate,"

according to whether or not one can express the work per second

of all three energies in terms of the kinematic and dynamic

variables which coexist in the differential cell, without intro-

ducing explicitly any new variables such as time, etc. Mechan-

ically undeterminate media will not be discussed here, while

for mechanically determined media one can derive from (1), (2)

and (3) an Equation of State which relates the coexisting kine-

matic and dynamic variables to one another, and which is formu-

lated in the scalars $\dot{W}$, $\dot{\phi}$ and $\dot{G}$ written as functions of

the said variables.

In order to emphasize the essential features of

such a scalar Equation of State, a number of assumptions

will here be introduced which greatly simplify the mathemat-

ical scheme, without reducing too badly the generality of

its applications,[1] _viz_.

Assumption 1. The medium is mechanically determinate
and exhibits the full symmetry of the Euclidean space in which
it is imbedded when subjected to mechanical tests in the initial
position. Thus, the medium in the initial position will exhibit
in such tests the symmetry of homogeneity (invariance against
all translation) and of isotropy and parity (invariance against
all rotations and mirror inversions).

(The full symmetry of the Euclidean space is postu-
lated only for the initial position and will not be maintained,
in general, in any other position.)

Assumption 2. The medium is incompressible.

Assumption 3. All the mechanical actions are carried
out at constant temperature, under homogeneous conditions of the
kinematic and dynamic variables, and in such a manner that the
resultant deformations are irrotational in the medium so that
the main axes of the deformation will extend in all positions
along one and the same triad of mutually perpendicular directions
fixed in the medium.

---

1. If and when necessary the assumptions can be discarded, one
by one, and the simple scheme discussed above can be used as a
base for the development of schemes of increasingly greater gen-
erality.

Assumption 4. The energy transformations in the medium are considered for a differentially small cell for which one can neglect the mass forces of gravity and inertia against the traction forces acting on the surface of the cell, since the former forces approach to zero with the third power of the length of the edge of the cell, while the latter approach to zero with the second power.

2. The Coexisting Kinematic and Dynamic Variables Defined as Coaxial Cartesian Deviatiors with Additive Laws of Superposition

Based on these assumptions, one can introduce for the coexisting kinematic and dynamic variables a system of Cartesian coordinates whose axes are fixed in the medium and extend in all positions along the three mutually perpendicular directions which mark for all the said variables the common directions of their main axes.[1] A restriction to Cartesian systems of coordinates will then be introduced and denoted by an asterisk attached to the number of the equation. This restriction allows us to disregard the distinction between true and pseudo-tensors and dispense with the stroke at the upper or lower index level

---

1. It follows from Assumptions 1 and 3 that the directions of the main axes of all the kinematic and dynamic variables must coincide in the medium.

which denotes for pseudo-tensors their character as volume capacities and volume densities. In short all the variables will be treated as true Cartesian tensors.

Furthermore, the kinematic and dynamical variables can now be so chosen that they all have additive superposition laws,[1] and that they are all Cartesian deviators, i.e., Cartesian tensors with a vanishing first invariant,[2] with uniquely defined time derivatives.[3]

---

1. The dynamic variables have always additive laws of superposition (see lecture on Dynamics) and for the kinematic variables it follows from Assumption 3 that for this special case one can introduce the logarithms of Θ as variables with an additive superposition law (see Part II of lecture on Kinematics).

2. The kinematic variables measured by the logarithmic scale functions have a first invariant equal to the change in volume and it follows from Assumption 2 that this change equals zero. The dynamic variables for an incompressible medium are only defined apart from an additive isotropic pressure and may therefore also conveniently be described by Cartesian tensors with vanishing first invariant.

3. For variables which have an additive superposition law the time derivatives are always uniquely defined (see Part II of lecture on the Kinematics).

The Cartesian Deviators and their time derivatives will be denoted for the kinematic variables[1] by

$$\overset{(\mu)}{\gamma} \text{ for } \mu = 0, 1, 2 \ldots \mu \text{ with}$$

$$\gamma = \ln \Theta \qquad \text{and} \qquad (4)*$$

for the coexisting dynamic variables by

$$\overset{(\nu)}{p} \text{ for } \gamma = 0, 1, 2 \ldots \gamma \text{ with}$$

$$p = P - 1/3\, P_I I \qquad (5)*$$

3. The Formulation of a Scalar Equation of State for

Mechanically Determinate Media

For mechanically determinate media one can now express the scalars $\dot{w}$, $\dot{\phi}$ and $\dot{G}$ in terms of the variables $\overset{(\nu)}{p}$ and $\overset{(\mu)}{\gamma}$, <u>viz</u>.,[2]

_____

1. It follows from Assumption 3 that in the chosen Cartesian system of coordinates the ante and postrotational deformation tensors are equal to one another (<u>i.e.</u>, $\Theta_a = \Theta_p = \Theta$), and are independent of the coordinates of the point considered so that it does not matter here whether the Lagrangian or Eulerian definition is used.

2. According to Assumption 4, one can neglect the mass forces, and hence express for media of all kinds the work of the external forces (per unit time and unit volume) by the double dot product of the Cartesian tensors of stress and deformation velocity. It then follows Assumption 3 that the work of the free and dissipated energy will here depend only on the coexisting kinematic and dynamic variables and their time derivatives as given above, since all the space derivatives are zero.

$$\dot{W} = p .. \dot{\gamma} \tag{6}*$$

$$\dot{\phi} = \frac{d}{dt} \phi(\overset{(v)}{p}\overset{(\mu)}{\gamma}) \tag{7}*$$

$$\dot{G} = \dot{G}(\overset{(v)}{p}\overset{(\mu)}{\gamma}) \tag{8}*$$

which substituted into (1) yields the Scalar Equation State

$$- p .. \dot{\gamma} + \frac{d}{dt} \phi(\overset{(v)}{p}\overset{(\mu)}{\gamma}) + \dot{G}(\overset{(v)}{p}\overset{(\mu)}{\gamma}) = 0 \tag{9}*$$

with the condition

$$\dot{G}(\overset{(v)}{p}\overset{(\mu)}{\gamma}) \geq 0 \tag{10}*$$

The Scalar Equation of State, and in fact all the equations (6)* to (10)*, may conveniently be expressed in terms of the three fundamental scalar invariants of $\overset{(v)}{p}$ and $\overset{(\mu)}{\gamma}$, or in terms of the three main values of these variables, with parameters which specify the two work functions $\dot{\phi}$ and $\dot{G}$, and so define the mechanical properties of the medium considered. One may then interpret the Scalar Equation of State by identifying the completely reversible work function $\dot{\phi}$ with the reversible work of a general elastic potential, and the completely irreversible work function $\dot{G}$ with the irreversible work of a general viscous dissipation,[1] so that the parameters which

---

1. The identification of the thermodynamical functions $\dot{\phi}$ and $\dot{G}$ with the work functions of certain mechanical energies should be regarded as a rough first approximation (see K. Weissenberg, Abh.d.Preuss, Ak.d.Wiss. (1931) Heft 2). It has recently been possible to develop a second approximation in connection with a new theory which takes into account the movements of dislocations, and the forming of cracks in phenomena of fatigue, nonhardening and dilatance.

specify the two functions will appear as mechanical properties described respectively by moduli of elasticity and viscosity. On the basis of the identification indicated above, a detailed investigation has been published in which the Scalar Equation of State was expanded into a cycle of theories. The general procedure of this expansion is given below.

4. An Expansion of the Scalar Equation of State into a Cycle of Theories

For a given medium the Scalar Equation of State may first be regarded as a differential equation with respect to time, whose integration provides the <u>Laws of the Aftereffects</u> of the coexisting kinematic and dynamic variables. One may then derive for the medium the <u>Laws of Elasticity, Viscosity and Dissipation</u> by resolving the said equation according to

$$
-\dot{W}:\dot{\phi}:\dot{G} = \begin{cases} -1:+1:\ 0 \\ -1:\ 0:+1 \\ \ \ 0:-1:+1 \end{cases} \qquad \text{respectively} \qquad (11)
$$

with the law of Dissipation further resolved into the <u>Laws of Retardation and Relaxation</u>, according to

$$
0 = \dot{W} = \phi_{..}\dot{\gamma} \quad \text{with} \begin{cases} \dfrac{d}{dt}\phi(\overset{(u)}{\gamma}) + \dot{G}(\overset{(u)}{\gamma}) = 0 \quad \text{for } \phi = 0 \\[2mm] \dfrac{d}{dt}\phi(\overset{(v)}{\phi}\gamma) + \dot{G}(\overset{(v)}{\phi}\gamma) = 0 \quad \text{for } \dot{\gamma} = 0 \end{cases} \qquad (12)
$$

and all the above-mentioned laws interrelated by <u>Maxwell's Theory of Elasticity and Dissipation together producing Viscosity.</u>

After carrying out the resolution as indicated above, one may close the cycle of theories through the Laws of Super-position by which three ideal media, obeying respectively the Laws of Elasticity, Viscosity, and Dissipation given in (11) are combined to give the original medium for which the Scalar Equation of State has been established.

Important special cases arise for two extreme classes of media, referred to respectively as ϕ -media, and γ -media, for which the scalars $\dot{\phi}$ and $\dot{G}$ are functions only of $\overset{(\nu)}{\phi}$, or only of the $\overset{(\omega)}{\gamma}$.

Seminar on Continuum Mechanics

Lecture VIII

Principles of Mechanical Properties in Continuous Media

Part II. The Strains and Stresses Defined in
a Common System of Reference for the Establishment of
a Tensorial Equation of State

A. Introduction

  1. A Common System of Reference

  2. Strains and Stresses to be Observed as Coexisting in
       the Terminal State on the Same Area of a Plane
       of any Orientation

  3. Invariance against All Changes of Coordinate Systems
       and of Scale Functions

B. Redefinition of the Coexisting Strains and Stresses in a
       Common System of Reference

  1. Formulation of the Definitions

  2. The Relations of the New Definitions to the Traditional
       Ones

C. Goniometry of the Newly Defined Strains and Stresses

D. The Choice of Scale Functions for the Measurement of the
       Strains and Stresses

E. A Model Technique for the States of Strains and Stresses

  1. Invariance against All Changes of Scale Functions

  2. The Anisotropic Similitude

Principles of Mechanical Properties in Continuous Media

Part II.   The Strains and Stresses Defined in

a Common System of Reference for the Establishment of

a Tensorial Equation of State

A.   Introduction

   1.   A Common System of Reference

      One of the most important but still controversial

fields of Continuum Mechanics is concerned with the development

of a Tensorial Equation of State which would correlate in the

differential cell of any given medium the traction forces and

the relative movements which coexist therein. It has already

been shown in one of the previous lectures how the mechanical

properties of the medium could be specified once such an

equation of state were given.

      The present attempt for the development of a Tensor-

ial Equation of State is based on the application of the gen-

eral transformation theory (see Lecture II). In this theory

a relativity principle has been established according to which

the definition of any quantity has physical significance only

relative to some chosen system of reference. It follows that

a correlation between any two quantities can have a physical

significance only if they are both defined relative to the same system of reference, as otherwise correlation would incorporate the completely extraneous relation between the two systems of reference used, and thereby confuse the issue. The systems of reference are here understood in the general sense of the word, so that each such system contains not only a system of space coordinates with a specified metric but also all elements, instruments, laws of calibrations, etc., which are necessary to observe and measure the quantities under consideration. Now, the Tensorial Equation of State attempts to correlate the coexisting traction forces and relative movements, but these two quantities have traditionally been defined with reference to observations made in different positions and on different geometric elements. In particular, the definition of the traction forces referred to observations made in the terminal position only, while the definition of the coexisting relative movements referred to observations made in the initial and terminal positions. Furthermore, the traction forces were observed on geometric elements chosen as areas of planes, while the coexisting relative movements were observed on distances between points.

2. Strains and Stresses to be Observed as Coexisting in
    the Terminal State on the Same Area of a Plane

In order to obtain physically significant correlation between the coexisting strains and stresses, it was necessary to depart from tradition and introduce a common system of reference with mutually adjusted definitions so that all the observations made at a point in the medium referred for both coexisting quantities to one and the same position, the terminal one, and to one and the same geometrical element, the area of a plane of any orientation centered on the point considered. In the common system of reference the traction forces and stresses will be defined as before, but the coexisting relative movements and deformations will have to be newly defined so as to be observable in the said system. The new definitions will then be related to the traditional ones by appropriately chosen scale functions, so that one can pass easily from one definition to the other in either way.

3. Invariance against all Changes of Coordinate Systems
    and of Scale Functions

The choice of a common system of reference is a necessary but not a sufficient condition for the establishment of a physically significant correlation between two quantities. A second equally important condition is that the correlation

should be defined in a manner which is invariant against all changes of the system of reference, i.e., invariant against all changes of the scale functions used in the measurement of the two quantities, as otherwise the arbitrary choice of the system of reference and particularly of the two scale functions will introduce a corresponding arbitrariness in the corellation between the two quantities, and thus destroy its physical significance.

The establishment of such an invariant correlation will be based on the coincidence of directions and/or on the equality of amounts, as both these conceptions have the required invariance against all changes. Once the said correlation has been established, it can be formulated in terms of any arbitrarily chosen scale functions, of the traction forces and stresses, and of the relative movements and deformations. It will be convenient to give the correlation first in its simplest algebraic form by means of scale functions specially selected for this purpose, and to pass from this form to a general one by expressing the specially chosen scale functions in terms of arbitrary ones.

B. Redefinition of the Coexisting Strains and Stresses in the
Common System of Reference

1. Formulation of the Definitions

At any point in the medium a differentially small cell will be considered, and in it the coexisting kinematic

and dynamic variables will be defined in a common system of reference. This system is so chosen that one can observe both coexisting quantities in one and the same position, the terminal one,[1] and on one and the same geometric element, the area of a plane of any orientation passing through the point considered.

To make such observations possible one imprints in the medium in its initial position certain marks which will be recognized at all instants of time, and in particular at the arbitrarily chosen instant in which the medium happens to be in the terminal position.

The marks in the chosen differential cell will be put on a pair of neighboring small test areas of equal size, $d\overset{o}{S}$ and $d\overset{o}{S}'$, say, which are parallel to one another, centered on the points $\overset{o}{O}$ and $\overset{o}{O}'$ respectively, and so oriented and located as to be perpendicular to the vector $d\overset{o}{s}$ that joins the two centerpoints.

In the terminal position one will recognize the marked test areas and their centerpoints, which will now serve as the common system of reference and which will be denoted by

---

1. The terminal position of the medium is considered because this is the general one which may be identified with any position of the medium at any arbitrarily chosen instant of time, including the initial position occupied at the instant $t = \overset{o}{t}$ as a special limiting case.

dB and dB', and by 0 and 0' respectively with the vector ds joining 0 to 0' . The relative movements, now referred to as tractional movements, will then be observed as the displacement of the area dB' relative to the test area dB, and the tractional forces as the forces dF exerted by the area dB' on test area dB. For the fractional movements one then defines the tensor of tractional deformation, or strain, as the operator which acts on the test area dB and transforms it into the vector ds, and for the tractional forces one defines the stress as the operator which acts on the same test area dB and transforms it into the force vector dF.

The algebraic calculations may be given in terms of tensors and pseudotensors, the latter being distinguished by a stroke at index level, indicating volume densities of tensors by a stroke at the upper level, and volume capacities by a stroke at the lower level. The test area will be given vector capacity dA_{k-} viz.,

$$dB^{ij} = dA_{k-} \tag{1}$$

where the pseudovector dA_{k-} is perpendicular to the test area and has a length equal to the size of that area. Any operator acting on that test area and transforming it into a contravariant vector ds^m or dF^m will then appear as a mixed tensor of third order, or a double contravariant tensor density of second order.

For greater convenience of calculation and without impairing the generality of the procedure, one normalizes the distance apart of the pair of neighboring areas with respect to the size of these areas by a normalization condition which postulates for the initial position

$$d\overset{o}{s}{}^{m} = d\overset{o}{B}{}^{ij} \overset{o}{I}{}_{ij}^{\cdot\cdot m} = d\overset{o}{A}{}_{k-} \overset{o}{I}{}^{km-} \tag{2}$$

where the indices ij, k refer to cyclical permutations of the indices 1, 2, 3 of the coordinate axes of a righthanded system, and where $\overset{o}{I}{}_{ij}^{\cdot\cdot m}$ and $\overset{o}{I}{}^{km-}$ denote the unit volume tensor of third order, and its contraction into a tensor density of second order in the coordinate system used in the initial position of the medium. The normalization condition (2) means that in the initial position of the medium there is a coincidence in length and orientation of the contravariant vector $d\overset{o}{s}$ and the covariant pseudovector $d\overset{o}{A}{}_{k-}$, the former representing the spacing bar vector between the areas $d\overset{o}{B}$ and $d\overset{o}{B}{}'$, and the latter representing the orientation and size of these areas.

With the above normalization condition one then defines in the terminal the strains and stresses as the operators acting on the test area according to the equations

$$ds^{m} = dB^{ij}T_{ij}^{\cdot\cdot m} = dA_{k-} s^{km-} \quad \text{and} \tag{3}$$

$$dF^{m} = dB^{ij}Q_{ij}^{\cdot\cdot m} = dA_{k-} p^{km-} \tag{4}$$

In view of the contraction of dB^{ij} into dA_{k-} of equation (1), one obtains for the strains and stresses

$$S^{km-} = T_{ij.}^{..m} - T_{ji.}^{..m} \quad \text{and} \tag{5}$$

$$P^{km-} = Q_{ij.}^{..m} - Q_{ji.}^{..m} \tag{6}$$

It will be noted that for the stress the definition here is identical with the traditional one (see lecture on Dynamics) but for the strain it is not. In the new definition the strain is conceived in the terminal position through the discrepancy there between the spacing bar vector ds^m and the representative pseudovector dA_{k-} of the test area. This discrepancy arose between the initial and terminal positions through the movement of the neighboring area relative to the test area, as there had been a coincidence of $d\overset{O}{s}{}^{m}$ and $d\overset{O}{A}_{k-}$ in the initial position according to the normalization condition (2). (In a rigid movement of the differential cell as a whole there would be no change in the spacing bar vector ($d\overset{O}{s}{}^{m} = ds^m$), and no change in the representative pseudovector of the test area ($d\overset{O}{A}_{k-} = dA_{k-}$), so that in this case there would be no discrepancy in the terminal position between ds^m and dA_{k-}.)

2. The Relations of the New Definitions to the Traditional Ones

For a quantitative relation between the new and the traditional definition of strains, one has to compare the equations (2) and (3) of this lecture with the equations given in

Part II of the kinematics. One then finds for the main values, S_i say, of the tensor density $\bar{S}$

$$S_{(i)} = \frac{\lambda_{(i)}^2}{\lambda_{(1)}\lambda_{(2)}\lambda_{(3)}} \qquad \text{for } i = 1,2,3 \qquad (7)$$

and for the directions of the main axes of $\bar{S}$ that the directions coincide with the direction of the post-rotational deformation tensor $\bar{p}$, which may be indicated by equating the appropriate unit vectors, $\epsilon_{(i)}$ say, viz.,

$$\epsilon_{(i)}(\bar{S}) = \epsilon_{(i)}(\bar{p}) \quad \text{for } i = 1,2,3 \qquad (8)$$

There is no need to establish for the stresses the corresponding relations as here the new and the traditional definitions coincide.

C. Goniometry of the Newly Defined Strains and Stresses

According to the new definitions one measures the coexisting strains and stresses by the tensor densities $\bar{S}^{mk}$ and $\bar{p}^{mk}$ (or by the tensors $T_{ij.}^{..m}$ and $Q_{ij.}^{..m}$), and one can best appreciate how well these measures are adjusted to one another by considering their goniometry. While in the traditional definitions there was a difference between the laws which govern the goniometry of the two coexisting quantities, this difference has now disappeared, and one finds for the new definitions that the two quantities obey the same goniometric laws. One finds in particular that they both have at every point in the medium round the full solid angle of directions

(a) the same transformation properties since they are both measured by tensor densities (or tensors) of the same order and the same symmetry

(b) the same laws of equilibrium for the two sides of a sectional test area of any orientation, and for the four test areas of the faces of a tetrahedron of any shape and any orientation since for a sectional test area one has

$$0 = ds^m + (-ds^m) = s^{km-} [dA_{k-} + (-dA_{k-})] \quad \text{and} \tag{9}$$

$$0 = dF^m + (-dF^m) = P^{km-} [dA_{k-} + (-dA_{k-})] \tag{10}$$

and for the four faces of a tetrahedron

$$0 = ds^m + ds^m_{(x)} + ds^m_{(y)} + ds^m_{(z)} =$$

$$s^{km-} [dA_{k-} + dA_{(x)k-} + dA_{(y)k-} + dA_{(z)k-}] \quad \text{and} \tag{11}$$

$$0 = dF^m + dF^m_{(x)} + dF^m_{(y)} + dF^m_{(z)} =$$

$$P^{km-} [dA_{k-} + dA_{(x)k-} + dA_{(y)k-} + dA_{(z)k-}] \tag{12}$$

because for the four faces of any tetrahedron one finds

$$0 = dA_{k-} + dA_{(x)k-} + dA_{(y)k-} + dA_{(z)k-} \tag{13}$$

where three faces of the tetrahedron are denoted by bracketed indices x, y and z, and the fourth face without bracketed index.

(c) the same laws of expressing the resolution into components parallel and perpendicular to the common test area of any orientation. One can resolve the tractional movements, as well as the coexisting tractional forces relative to the

plane of the common test area of any orientation into three
mutually perpendicular components. One component will be
normal to the plane of the area, a second will be tangential
to it, and a third will extend along a direction perpendicular
to the other two. The tangential direction will be chosen
for the strains along the projection of the tractional move-
ment of the parallel neighboring area relative to the test
area, and for stress along the projection of the tractional
force on the test area.

For the algebraic calculations the three components
will be referred to as the "normal," "tangential" and "cross"
components and denoted by a bracketed index N, Tg, and C re-
spectively. For the convenience of working with quantities
of finite size only, one divides the equations (3) and (4)
by the infinitely small size dA say, of the test area

$$dA \quad = \quad dA_{k-}.dA^{k-})^{1/2} \tag{14}$$

This division will not affect the tensor densities s^{km-} and
p^{km-} (since both sides of the equations (3) and (4) will be d
divided by the same factor), but it will reduce all the infini-
tesimally small quantities dA_{k-}, ds^m and dF^m to unit area. One
then finds for the resolution

$$\frac{dA_{k-}}{dA} \ s^{km-} = \frac{ds^m}{dA} = \frac{ds^m_{(N)}}{dA} + \frac{ds^m_{(Tg)}}{dA} + \frac{ds^m_{(C)}}{dA} \quad \text{and} \tag{15}$$

$$\frac{dA_{k-}}{dA} \ p^{km-} = \frac{dF^m}{dA} = \frac{dF^m_{(N)}}{dA} + \frac{dF^m_{(Tg)}}{dA} + \frac{dF^m_{(C)}}{dA} \tag{16}$$

which gives for the normal components

$$\frac{ds\,^{m}_{(N)}}{|dA|} = \left(\frac{dA_{u-}}{|dA|}\,\frac{dA_{k-}}{|dA|}\,s^{ku-}\right)\frac{dA^{m-}}{|dA|} \qquad \text{and} \qquad (17)$$

$$\frac{dF\,^{m}_{(N)}}{|dA|} = \left(\frac{dA_{u-}}{|dA|}\,\frac{dA_{k-}}{|dA|}\,p^{ku-}\right)\frac{dA^{m-}}{|dA|} \qquad\qquad (18)$$

and for the tangential components

$$\frac{ds\,^{m}_{(Tg)}}{|dA|} = \frac{dA_{k-}}{|dA|}\,s^{km-} - \left(\frac{dA_{u-}}{|dA|}\,\frac{dA_{k-}}{|dA|}\,s^{ku-}\right)\frac{dA^{m-}}{|dA|} \qquad (19)$$

$$\frac{dF\,^{m}_{(Tg)}}{|dA|} = \frac{dA_{k-}}{|dA|}\,p^{km-} - \left(\frac{dA_{u-}}{|dA|}\,\frac{dA_{k-}}{|dA|}\,p^{ku-}\right)\frac{dA^{m-}}{|dA|} \qquad (20)$$

and finally for the components

$$\frac{ds\,^{m}_{(C)}}{|dA|} = 0 \qquad\qquad (21)$$

$$\frac{dF\,^{m}_{(C)}}{|dA|} = 0 \qquad\qquad (22)$$

The algebraic calculations given above can be re-placed by a graphical construction which has been known as the Mohr circle diagram for the resolution of the tractional forces. In this diagram the main values $P_{(i)}$ (for $i = 1,2,3$) of the stress, as measured by p^{km-}, are extended along the x axis, and half circles drawn over the distances between them. The normal and tangential components are then read off as the x and y coordinates of a point representing the orientation of the test area. As the formulae for the resolution of the

tractional forces in terms of P^{km-} are now analogous to those

of the resolution of the tractional movements in terms of the

S^{km-} one can use a Mohr circle diagram also for the latter by

drawing half circles over the main values $S_{(i)} = \dfrac{\lambda_{(i)}^2}{\lambda_{(1)} \lambda_{(2)} \lambda_{(3)}}$

(for i = 1,2,3) and then proceeding as above.

D. The Choice of Scale Functions for the Measurement of the

Strains and Stresses

The choice of any particular scale function for the

measurement of the strains and stresses is arbitrary, and

should be adjusted so as to offer the best advantages for the

particular problem under consideration. It has not been

found possible to find scale functions which are advantageous

to all the problems one has to deal with in Continuum Mechan-

ics. Hence, for different problems different choices will be

made, and one has to make the appropriate changes in the scale

functions as one proceeds from one problem to another. This

can be done for the strains with the help of equations (7) and

(8), while for the stresses there is no need to do this as the

same measure is used in the various applications.

At present we are concerned with the problem of

finding an Equation of State which will relate the coexisting

(tractional) strains and stresses to one another, and for this

purpose one has to use the equations (1) to (22). It will be

advantageous for this purpose to have the scale functions of

the two coexisting quantities so adjusted to one another that

they obey at every point of the medium the same goniometric

laws round the full solid angle, and that they involve only

the powers up to the first degree and are in this sense quasi-

linear or, more exactly "tensor-linear." It will be noted

that the scale functions S^{km-} and P^{km-} have the advantage of

being so adjusted, and only involved to powers up to the first

degree so that all the equations (1) to (22) are tensorial-

linear.

Without losing any of the above-mentioned advantages,

one can now introduce for the coexisting strains and stresses

new scale functions which are related to S^{km-} and P^{km-} by a

tensorial-linear relation, viz.,

for the strain

$$= H(S^-) = KS^{km-} + LI^{km-} \qquad \text{and} \qquad (23)$$

for the stress

$$G^- = G(P^-) = MP^{km-} + NP^{km-} \qquad (24)$$

where I^{km-} denotes the volume density of the unit tensor, and

K, L and M, N denote mutually adjusted scalar quantities which

may be constants, or any functions of scalar variables and/or

of the scalar invariants of S^{km-} and P^{km-}.

There will be no need to change the scale function

P^{km-} for the stresses, but sometimes it will be useful to do

so for the strains and use here a scale function which has zero

value for no strain. Such a scale function can be constructed
as

$$H^{km-} = S^{km-} - \overset{O}{S}{}^{km-} = S^{km-} - I^{km-} \tag{25}$$

as in the initial position the value $\overset{O}{S}{}^{km-}$ is always equal to
I^{km-} according to (7). The use of this scale function of the
strains does not require an adaptation of the scale function
P^{km-} for the stress since in the initial position $\overset{O}{P}{}^{km-}$ is zero,
and hence

$$G^{km-} = P^{km-} - \overset{O}{P}{}^{km-} = P^{km-} \tag{26}$$

When one proceeds from the privileged group of scale
functions to any arbitrarily chosen ones,

$$C^- = C(S^-) \tag{27}$$

$$D^- = D(P^-) \tag{28}$$

in terms of convergent power series, involving powers of degrees
higher than one in the S^{km-} and P^{km-} and thereby making the
equations (1) to (22) nonlinear. Such a procedure would there-
fore not be advantageous here but may have to be adopted when
one deals with other problems which may require the use of
scale functions other than those of the privileged group. In
these cases one may reduce the power series and obtain a quasi-
quadratic or, more exactly, a tensorial quadratic expression
by replacing all powers of third and higher degrees by powers
of second and lower degrees in accordance with the Cayley

Hamilton equation.[1] An example of this will be discussed in the next section.

E. A Model Technique for the States of Strains and Stresses

    1. Invariance against all Changes of Scale Functions

        The traditional model technique in Continuum Mechanics is based on the use of dimensionless quantities, i.e., of quantities which are invariant against changes of the units of mass, time, and length, and two systems are regarded as similar (one serving as a model for the other) if their characterizations in terms of all the dimensionless quantities coincide. This procedure works satisfactorily when one wants to compare states which are different in their dimensions of mass, time and space, but becomes ineffective when one wants to compare states which are different in some dimensionless quantities, such as the strains. In these cases one can not apply the model technique as the similitude defined above requires an identity of the dimensionless quantities. It

---

1. The Cayley Hamilton equation applies to any tensor of second order, X say, and relates the successive powers of X as

$$X^3 - X_I X^2 + X_{II} X - X_{III} 1 = 0$$

where X_I, X_{II} and X_{III} denote the three scalar invariants of X

will be seen that the concept of similitude can be generalized so as to apply also to states with different amounts of dimensionless quantities, and, in particular, to states with different amounts of strains.

One may be tempted to generalize the concept of similitude by that of algebraic proportionality of the dimensionless quantities but such a generalization has a physical significance only with regard to the particular scale function with which the dimensionless quantity happens to be measured, and two states which appear similar with respect to one scale function would be dissimilar with respect to another (e.q., two states of strains which appear similar when measured in percent of elongation would appear dissimilar when measured in logarithms of the elongation).

In order to apply a model technique to states with dimensionless quantities of different amounts, it is necessary to generalize the definition of similitude in such a way that it is invariant against all changes of the scale functions, and thus capable of a physically significant interpretation. Such a generalized invariant definition can be formulated by way of an identity of certain angles which are associated with the quantities considered. Once the invariant definition of the generalized similitude is given, one can apply it equally to all quantities, with or without dimension, and one can

formulate it in terms of any arbitrarily given scale functions of the said quantities. Moreover, one can then also find a privileged group of scale functions in which the generalized similitude appears in the form of an algebraic proportionality.

2. The Anisotropic Similitude

The quantities with which we are here concerned in the development of a model technique are the coexisting strains and stresses, of which the former is dimensionless while the latter has the dimensions of $[m' \ t^{-2} \ l^{-1}]$, and the associated angles can be obtained by considering for a test area of any orientation the three mutually perpendicular directions of the normal, tangential and cross components. It is easily seen that all the angles involved in this consideration are invariant against all changes in the scale functions with which the coexistent strains and stresses are measured. A generalized concept of similitude can now be defined in an invariant manner for two states of strains, and separately for two states of stress, and again separately for a state of strain and a state of stress, by postulating identity of the angular distributions of the various components with respect to every test area of a given orientation. The generalized similitude will be referred to as anisotropic similitude since the said angular distributions are unaffected by strains and stresses of isotropic symmetry, and determined completely by the anisotropy of these quantities.

One can formulate the anisotropic similitude in algebraic form first in terms of the particular scale functions S^{km-} and P^{km-} by reference to equations (19) to (22), and then in terms of any arbitrarily chosen scale functions C^{km-} and D^{km-} by inverting equations (27) and (28), and expressing S^{km-} and P^{km-} in terms of C^{km-} and D^{km-}.

Using the particular scale functions S^{km-} and P^{km-} one finds then from (19) to (22) in an invariant notation that anisotropic similitude exists

for two states of strain S^- and S'^- say, if

$$XS^- + YS'^- + ZI^- = 0 \tag{29}$$

for states of stress P^- and P'^-, say, if

$$KP^- + LP'^- + MI^- = 0 \tag{30}$$

and for a state of strain S^- and a state of tractional stress P^- if

$$UP^- + VS^- + WI^- = 0 \tag{31}$$

where I^- is the volume density of the unit tensor, and the coefficients X, Y, Z, K, L, M, and U, V, W denote scalars which need not be constants but may be scalar functions of any scalar variables including time, temperature, and the scalar invariants of the strains, the stresses and their derivatives in space and time. The equations (25) to (31) can be written in various other forms which are all equivalent to one another, and which are most convenient for various purposes. One such form is

obtained by eliminating the scalar coefficients, and this brings out clearly the character of the anisotropic similitude in terms of the particular scale function as an algebraic proportionality of certain differences. The elimination of the scalar coefficients can be carried out by subtracting from the equations (29) to (31) other equations derived from them by successive rotations which produce cyclical permutations of the coordinate axes. These rotations leave the scalar coefficients unchanged (because of their isotropic symmetry) while the components of the strains and stresses suffer cyclical permutations of their indices. By considering then the ratios of the differences of the components one obtains equations which are free from the scalar coefficients. The results can conveniently be written in a restriction to Cartesian systems of coordinates, where the restriction is indicated by an asterisk to the number of the equation, and where one can disregard the distinction between the true tensors and tensor densities and hence disperse with the distinctive stroke at index level of the said densities. One then finds from (22) (23) and (24)

$$(s^{au}-s^{bv}):(s^{bv}-s^{cw}):(s^{cw}-s^{au}) = (s'^{au}-s'^{bv}):(s'^{bv}-s'^{cw}):(s'^{cw}-s'^{au}) \quad (32)*$$

$$(p^{au}-p^{bv}):(p^{bv}-p^{cw}):(p^{cw}-p^{au}) = (p'^{au}-p'^{bv}):(p'^{bv}-p'^{cw}):(p'^{cw}-p'^{au}) \quad (33)*$$

$$(p^{au}-p^{bv}):(p^{bv}-p^{cw}):(p^{cw}-p^{au}) = (s^{au}-s^{bv}):(s^{bv}-s^{cw}):(s^{cw}-s^{au}) \quad (34)*$$

or in terms of the main values

$$(S_{(1)}-S_{(2)}):(S_{(2)}-S_{(3)}):(S_{(3)}-S_{(1)}) =$$

$$(S'_{(1)}-S'_{(2)}):(S'_{(2)}-S'_{(3)}):(S'_{(3)}-S'_{(1)}) \qquad (35)*$$

$$(P_{(1)}-P_{(2)}):(P_{(2)}-P_{(3)}):(P_{(3)}-P_{(1)}) =$$

$$(P'_{(1)}-P'_{(2)}):(P'_{(2)}-P'_{(3)}):(P'_{(3)}-P'_{(1)}) \qquad (36)*$$

$$(P_{(1)}-P_{(2)}):(P_{(2)}-P_{(3)}):(P_{(3)}-P_{(1)}) =$$

$$(S_{(1)}-S_{(2)}):(S_{(2)}-S_{(3)}):(S_{(3)}-S_{(1)}) \qquad (37)*$$

It is seen from equations (35)* to (37)* that the above invariantly defined anisotropic similitude shows up in the Mohr circle diagrams of the strains and stresses as an ordinary geometric similitude of the said diagrams provided only that the diagrams are constructed for the main values of the particular scale functions S^{km-} and P^{km-}, or for the main values of any scale functions of the privilege group defined in equations (23) and (24).

It is also seen from the equations (29) to (37)* that the group of scale functions which was privileged with respect to the equations (1) to (22) is also privileged with respect to equations (29) to (37)*, as all these equations appear in tensor-linear forms when expressed in scale functions of the privileged group. However, when one introduces according to (27) and (28), arbitrarily defined scale functions

C^- and D^- which do not belong to the privileged group, then all the equations (1) to (22) and (29) to (37)* become tensor-quadratic, as one has to substitute for s^{km-} and p^{km-}.

The quadratic expressions

$$s^{km-} = c_0 I^{km-} + c_1 C^{km-} + c_2 C^{kl-} C_{1r-} g^{rm-} \qquad \text{and} \qquad (38)$$

$$p^{km-} = d_0 I^{km} + d_1 D^{km-} + d_2 D^{kl-} D_{1r-} g^{rm-} \qquad (39)$$

are derived from the inversion of (27) and (28), where the various coefficients c and d are scalars, and scalar functions of any scalar variables, including the scalar invariants of C^{km-} and D^{km-}, while g^{rm-} denotes the volume density of the fundamental metric tensor g used in the system of coordinates.

One finds in particular for the anisotropic similitude of two states from (29) (30) and (31) three tensor-quadratic equations, which are all analogous to one another so that it will suffice to give here one of them, namely, the one referring to anisotropic similitude of a state of strain with a state of stress. This equation may be written in terms of invariant notation as

$$U (d_0 I^- + d_1 D^- + d_2 D^- D_- g^-) + V (c_0 I^- + c_1 C^- + c_2 C^- C_- g^-) + W I^- = 0 \qquad (40)$$

where U, V and W are scalars, and arbitrary functions of any scalar variables, including the scalar invariants of C^- and D^-, etc., while the various coefficients c and d are scalars defined by (38) and (39).

Seminar on Continuum Mechanics

Lecture IX

Principles of Mechanical Properties of Continuous Media

Part IV. The Tensorial Equation of State Derived by Way of
an Analytical Procedure

A. Introduction

   1. Different Approaches to a Tensorial Equation of State

B. Analysis of the Correlation between Coexisting Strains and
Stresses

   1. Analysis in Time and Space

   2. Various Formulations of the Principle

   3. Prehistory and Elastic Recovery

   4. Theory of Elasticity

   5. Application to Simple (Two-Dimensional) Laminar Shearing
Motions

Seminar on Continuum Mechanics

Lecture IX

Principles of Mechanical Properties of Continuous Media

Part IV.   The Tensorial Equation of State Derived by Way of
an Analytical Procedure

A. Introduction

1. Different Approaches to a Tensorial Equation of State

When one tries to derive for a given continuous medium

a tensorial equation of state by correlating in a differential

cell the coexisting strains and stresses (defined there in a

common system of reference), one has to expect that this equa-

tion will differ from medium to medium, and even for the same

medium with the time and with the particular mechanical actions

applied. It is in these differences that the various phenomena

of flow and the various associated mechanical properties of

continuous media will show up in all their complexities, re-

sulting from all possible linear and nonlinear correlations

between the coexisting strains and stresses, their derivatives

in time and space, and other variables which may enter the

equation explicitly, such as time, temperature, boundary con-

ditions, etc.

In order to establish here a principle that would

apply to all the various media and conditions, one has to find

some feature which is common to all of them, and which has

physical significance in the sense that it is invariant against all coordinate transformations and against all changes of the scale functions. If such a feature can be found, then the required principle can be derived from it, and formulated as a tensorial equation of state of physical significance. This equation will express the feature in which all the media behave alike, and will at the same time incorporate sufficient unspecified parameters to allow the taking into account of all the differences and complexities which have to be expected in the mechanical behavior of media, as mentioned above.

In order to avoid, as far as possible, a reduction in the generality, we shall retain here only the assumption 1 of those made in the discussion of the thermodynamics of the energy transformations with a scalar equation of state (see lecture VIII), and discard all the other assumptions. The assumption 1 is retained here because it reduces the generality only slightly while it greatly simplifies the considerations and brings out more clearly the essential points.[1] For ease of reference the assumption will here be stated again, viz.,

Assumption IX.1. The media here considered will be restricted to those which in mechanical tests exhibit in the initial position but not necessarily in any other position the

---

1. It has been possible to discard this assumption as well and to establish a principle which applied to all media, whatever their symmetry may be.

full symmetry of the Euclidean space in which they are imbedded, i.e., the symmetry of homogeneity, isotropy and parity.

The establishment of a common feature in the mechanical behavior, and of a corresponding principle and tensorial equation of state can be achieved in a variety of ways leading to different formulations which, however, are all mutually consistent and interrelated. Only two of these ways will be selected here for discussion. The first one, based on an analytical procedure, will be reviewed in the present lecture, while the second one, based on a synthetic procedure, will be discussed in the following lecture.

B. Analysis of the Correlation between Coexisting Strains and Stresses

1. Analysis in Time and Space

At the start one may consider a medium of any kind which occupies its initial position at an instant $t = \overset{o}{t}$, and is moved thereafter under given conditions to a terminal position occupied at an instant $t = t$, say. In the said terminal position attention will be concentrated on a point O of the medium which is marked at the center of small test areas of all orientations. On each of these test areas one can observe the coexisting tractional forces and tractional movements (see preceding lecture), but the latter are observable there only in the form of the total tractional displacement from the

initial to the terminal position without any indication of the
path by which the medium travelled between the two positions.
In the special case of an ideally elastic medium, one can dis-
regard the path taken, but for the deduction of a general
principle applicable to a medium of any kind, it will be nec-
essary to take this path into consideration and to regard the
tractional forces on every test area dependent on the whole
prehistory of the tractional movements there. Resolving then
for every test area the tractional forces into the three mutu-
ally perpendicular components normal, tangential and crosswise
to the test area, one has to correlate there the directions and
amounts of each of these components with the prehistory of the
directions and amounts of all the three components of the trac-
tional movements and assess the relative magnitudes of the
terms entering in the correlation.

It will be assumed that this assessment can be made
in successive degrees of approximation, and that the resultant
correlation can be resolved into two mutually independent parts,
one dealing only with the directions, and the other only with
the amounts of the components. It will be further assumed that
the said resolution and assessments are applicable to all media
and conditions and so establish for them a common feature of
mechanical behavior.

For the part dealing with the correlation of the di-
rections only, one can make an assessment by postulating that

the direction of any one component of the traction force de-
pends strongly on the prehistory of the direction of the cor-
responding component of the tractional movement while only
weakly, or not at all, on the prehistory of the directions
and amounts of the two other components, so that these can
be neglected in a first approximation, and accounted for only
in a second and higher degree of approximation. The said as-
sessment may conveniently be introduced in the form of an
assumption, viz.,

Assumption IX.2. In any terminal position of the
medium, one may consider at every point a given test area and
expect to find there the normal, tangential and cross compo-
nents of the traction force in or near the ranges covered
there by the corresponding components of the tractional move-
ments during the whole prehistory from the initial to the
terminal position.

In a first approximation this means that a vector
marking the direction of the tangential component of the trac-
tion force for a given test area must be linearly dependent
on the set of vectors drawn in the various directions which
the tangential components of the tractional movements occupy
in the said test area in the course of the prehistory from the
initial to the terminal position. Considering then that in
the given test area the cross directions are already deter-
mined by the tangential ones, and that the normal directions

are the same for the tractional forces and the tractional move-
ments, one finds that a vector drawn in the direction of the
total tractional force must be linearly dependent on the set
of vectors drawn in the various directions which the normals
to the test area occupy in the medium in the course of the
prehistory from the initial to the terminal position. In the
second and higher degrees of approximations one has then to
modify the linear dependency by appropriate correction terms.[1]

Passing from the part of correlation dealing with
the directions of the components of the tractional forces
and tractional movements to the complementary part dealing
with the amounts of the said components, one finds for an
assessment a difficulty which has to be overcome. It arises

---

1. The assumption has been formulated above in exactly the
same way for the directions of all three components in order
to show its self-consistency in the sense that the same rule
applies to each of the said directions. It would have suf-
ficed, however, to consider only for the directions of the
tangential components since for the directions of the normal
components the assumption is always true, but trivial, since
in every position the normal direction of the tractional forces
must coincide with the normal direction of the tractional move-
ments, and for the directions of the cross components the as-
sumption does not contain anything new since these directions
are already defined by the other two components.

out of the fact that some media (the incompressible ones)

have at each point of the medium tractional forces which are

only determined by the tractional movements there, apart from

an additional isotropic pressure which remains here unspeci-

fied. For this reason an assessment will be made at the point

not for one test area of given orientation, but for a compari-

son of the amounts observed on two test areas of different

orientations. In such a comparison the traction forces and

all their components will be reduced to unit area, since then

the observations made on the two different test areas will be

comparable, and the unspecified isotropic pressure cancels

out at each point (but not in a comparison between two differ-

ent points). The assessment can then be made in accordance

with the following assumption.

Assumption IX.3. For a given point in the medium,

one may observe the traction forces in the terminal position

on two test areas of different orientations, and one will then

find in a first approximation that these forces have per unit

area the same normal (or tangential, cross) components if the two

test areas had per unit area the same prehistory (from the

initial to the terminal position) in respect of the amounts

and directions of the normal (or tangential, cross) components

of the tractional movements there. In second and higher de-

grees of approximation, deviations may occur as one may have

to take into account the prehistories of the two test areas

(after reduction to unit size) in respect to all three compo-

nents of the tractional movement where one assesses for the

tractional forces the equality of any one of its components.

However, such deviations are expected to be of small per cent

values.[1]

---

1. The self-consistency of the assumption has been again put
in evidence by a formulation which applies equally to all three
components. However, it may be noted that the assumption is true
but trivial for the cross components of the tractional forces
since they are by definition equal to one another (and zero) on
test areas of all orientations, and hence certainly equal on
test areas which had the same prehistory in respect to the direc-
tions and amounts of the cross components of the tractional move-
ments there. For the tangential components of the tractional
forces, the restrictions for an equality of their amounts are
very severe as they require for the two test areas the same pre-
history both in directions and in amounts of the tractional move-
ments there. This means that in the general case of an arbi-
trary prehistory there is hardly any possibility of using the
assumption for the said tangential components, unless and until
the special cases are considered in which the prehistory can be
disregarded, as, e.g., for media and conditions for which the pre-
history is replaced by an "elastically recoverable" tractional
movement (see later chapter of this lecture). Finally, for the
normal components of the tractional movement, the assumption is
always of importance since here the two test areas will have auto-
matically the same prehistory in direction as the normal direc-
tion of a test area is uniquely defined (while the tangential
direction is not).

The assumptions IX.2. and IX.3. express common features in mechanical behavior with the required invariance against coordinate transformations as well as against changes in the scale functions, and it can also be shown that they are consistent with one another.[1] Hence altogether, they are suited to derive from them a principle which establishes the common features in mechanical behavior, and thus can be used as a basis for a tensorial equation of state of general applicability. In such a derivation one can evaluate the validity of the assumptions with various degrees of approximations, and in each such approximation one can obtain different but mutually consistent formulations of the principle and the associated tensorial equations of state.

The evaluation of the validity of the assumptions in approximations of second and higher degree would require more time than is available here, and the discussion in this lecture will therefore be limited to an approximation of first degree only, in which the assumptions are simplified by deliberately neglecting in them the terms of higher degree against

---

1. A proof for the consistency has been published by P.U.A. Grossman in Koll. Zs. 174, 97 (1961). The proof there was restricted to ideally elastic media but has meanwhile been generalized so as to apply to media and conditions of all kinds.

those of first degree. The same will be done for the various formulations of the principle and the tensorial equation of state, and for ease of reference and discussion the formulations will be given as if they were exact, it being tacitly understood that the phrase "in a first approximation" should be added to each equation.

2.   Various Formulations of the Principle

When assumptions IX.2. and IX.3. are formulated in approximations of first degree, they can be put into algebraic forms which are invariant against all coordinate transformations and changes of scale functions by using Gram's determinant for expressing linear dependence and equations (17) to (22) of lecture VIII for expressing the various components of the tractional movements and tractional forces.

In particular one deduces from assumption IX.2. with reference to one test plane, viz.,

Principle (Formulation 1): During the prehistory from the initial to the terminal position one may mark in the medium vectors normal to a given test area at various instants $t = \overset{o}{t}, \overset{a}{t}, \ldots \overset{k}{t} \ldots t$, at which the medium occupies the positions (o), (a) ... (k) ... (l). One then finds in the terminal position imprinted in the medium a set of vectors which will outline in the medium a space of one or two or three dimensions,

and this space, whatever its dimensions may be, will always
contain the vector which indicates the direction of the trac-
tion force in the terminal position.

The principle can be formulated as a tensorial equa-
tion of state with the help of the Gram determinant,[1] G say.
This determinant formed from any given set of vectors has a
rank R_G which is equal to the number of vectors that are line-
arly independent from one another, and therefore always equal
to the number of dimensions of the space outlined by the set
of vectors considered. The derivation of the tensorial equa-
tion of state is based on the following considerations.

A vector which has been marked in the medium at the
instant $t = \overset{o}{t}$ perpendicular to the given test area in the in-
itial position will be there, along $d\overset{o}{s}$, and will occupy in the
medium in the terminal position a direction along ds which will,
in general, be inclined to the test area (see lecture VIII,
equations (2) and (3)). Similarly, one will find in the me-
dium in the terminal position inclined vectors along $ds(\overset{a}{t})$...
$ds(\overset{k}{t})$, say, resulting from vectors which at the instants $t = \overset{a}{t}$,
... $t = \overset{k}{t}$, etc., had been drawn perpendicular to the test area

_____

1. Gram's determinant for a set of vectors a,b,c, say is given by

$$G\ (a,b,c)\ =\ \begin{Vmatrix} a.a & a.b & a.c \\ b.a & b.b & b.c \\ c.a & c.b & c.c \end{Vmatrix}$$

in the intermediate positions (a) ... (k), etc. Finally, there
will be in the medium in the terminal position a vector which
had been drawn in that position and thus is perpendicular to
the test area along dA. It will be convenient to denote by
$s(t)$ for $t = \overset{o}{t}, \overset{a}{t}, \ldots \overset{k}{t}$, etc., the unit vectors along the in-
clined directions and by n the vector which is perpendicular.
One then finds for the tensorial equation of state

$$R_G [F, s(t), n] = R_G [s(t), n] = \begin{cases} 3 \text{ for } G\ s(t),n & \neq 0 \\ \left.\begin{matrix} 2 \\ 1 \end{matrix}\right\} \text{for } G\ s(t),n & = 0 \end{cases} \quad (1)$$

where F indicates the vector of the tractional force per unit
size of the test area.

The principle and equation (1) are certainly true,
but trivial, for $R_G = 3$ as this only means that the direction
of F lies somewhere in three-dimensional space. This leaves
for discussion the two degenerate cases, and here the validity
of the principle and of equation (1) can be proved for $R_G = 1$,
since in this case the tractional movements have occurred in
a direction normal to the test area, i.e., along a main axis
of the strain, and it follows from the isotropic symmetry of
the medium in the initial position that the tractional forces
too must then extend normal to the test area, i.e., along a
main axis of stress. Finally the degenerate case of $R_G = 2$
contains the essence of the principle and equation (1), and
this case cannot be proved theoretically (except by reference

to the assumption IX.2). It follows from (1) that for

$R_G = 2$, the tractional force must lie in the plane (space of

two dimensions) which contains the normal to the test area

and the direction s which indicates the tractional movement

of the parallel neighboring area relative to the test area.

Proceeding then from the consideration of one test

area to the consideration of two test areas of different orien-

tation, one deduces from assumption IX.3. for the normal com-

ponents[1]

Principle (Formulation 2): At any point in the me-

dium one finds in the terminal position for two test areas of

different orientations that the amounts per unit area of the

normal components of the tractional forces will be equal if

the two test areas had per unit area the same prehistory (from

the initial to the terminal position) of the amounts of the

normal components of tractional movements. Denoting by n_

and m_ the vector capacities which represent the two differ-

ently oriented test areas of unit size in the terminal posi-

tion, one finds from (17) and (18) of lecture VIII a ten-

sorial equation of state, viz.,

---

1. The formulation for the other components will be postponed

until the special cases are considered for which the prehistory

of the tractional movements can be replaced by elastically re-

coverable tractional movements (see footnote to assumption IX.3).

$$(n\_ \cdot \bar{P} \cdot n\_) - (m\_ \cdot \bar{P} \cdot m\_) = K \cdot \left[(n(t)\_ \cdot \bar{S} \cdot n(t)\_) - (m(t)\_ \cdot \bar{S} \cdot m(t)\_) \right] \quad (2)$$

$$\text{with} \quad \bar{P} = d_o \bar{I} + d_1 \bar{D} + d_2 \bar{D} D\_ \quad (3)$$

$$\bar{S} = c_o \bar{I} + c_1 \bar{C} + c_2 \bar{C} C\_ \quad (4)$$

where $\bar{C}$ and $\bar{D}$ denote arbitrary scale functions of the stresses
and strains, K denotes an unspecified factor which may depend
on all the parameters that characterize the conditions of the
experiment under consideration, and $n(t)$ and $m(t)$ denote the
quantities $n\_$ and $m\_$ as functions of time.

The above formulation of the principle has the re-
quired invariance and has the advantage of allowing exceedingly
rigorous experimental tests, in which systematic variations
are made of the kind of material, and of all the conditions
of the experiments, since the principle claims to apply to all
kinds and conditions.

Particularly suited for such tests are movements of
simple (two-dimensional) laminar shear, since for these move-
ments the second formulation of the principle claims for each
instant of time and each point in the medium that the normal
pressures per unit area should then and there be equal on all
test areas which are parallel to the direction of shear, since
all these areas have suffered per unit area the same prehistory
in respect to their normal displacements from their respective
parallel neighbors, all these normal displacements being zero
throughout the whole movement of the medium from the initial
to the terminal position considered.

Denoting then in a Cartesian system of coordinates
by indices 1, 2 and 3, respectively, the direction parallel
to the lines of flow, the direction normal to the shearing
plane, and the direction perpendicular to the other two, one
therefore has here

$$p^{22} = p^{33} \tag{5}*$$

irrespective of the boundary conditions, the mechanical prop-
erties of medium, the correlation between the stresses and
strains, and their derivatives in time and space, irrespective
too of whether or not there is a coincidence or an angular
deviation between the main axes of the stresses and strains,
and/or of their derivatives, and finally irrespective of changes
in temperature, etc.

3. Prehistory and Elastic Recovery

There are many phenomena which illustrate the influ-
ence of the prehistory on the mechanical behavior of materials.
When we speak hereafter of an elimination of the prehistory,
it does not mean that these phenomena are to be neglected.
What it does mean is that in the formulation of the principle,
we attempt to make no explicit reference to the prehistory
(which is sometimes unknown) and refer instead to results of
experiments which can be performed in the terminal position
without any knowledge of how this position has been reached.

A clear case is provided by ideally elastic media,
i.e., by media which possess a strain potential. According
to the theory of elasticity such media have a stress-strain
relation which can be specified in the terminal position with-
out any knowledge of how that position was reached. The stress
in the terminal position can be measured directly by way of
the applied forces, and the corresponding strain can be found
by releasing the forces, and observing the elastic recovery of
the medium in its return to an unstressed state. The strain
is then measured as between the position occupied in that
state and the terminal position. There is no need to know
here the exact path of the movement, or the velocities, accel-
lerations, etc., involved since the existence of a strain po-
tential indicates that the stress-strain relation depends only
on the total strain between the positions of the stressed and
unstressed states irrespective of all other conditions.

For media of a general kind which have some sort of
elasticity but are not ideally elastic, the position is more
complicated, and not quite clarified yet. There are many phe-
nomena of various kinds which are connected with the elastic
energy stored in such media. Two such phenomena are connected
with the dissipation of the stored energy under conditions of
zero external work (i.e., at $P^- \ldots \dot{S}_- = 0$) according as one
observes the "relaxation" of stress P^- for $\dot{S}_- = 0$, or the "re-
tardation" of strain for $P^- = 0$. One can use either of these

phenomena for an assessment of the elastic parts of the stresses and strains. Moreover, one can superimpose in the terminal position vibrations of stresses and strains, and resolve the resultant complex modulus into moduli of viscosity and elasticity. The best procedure has not yet been settled, but provisionally, the following suggestion has been made.

The whole stress should be regarded as elastic, and an "elastically recoverable strain" should be defined so as to correspond to that stress. In previous publications it was assumed that this strain is approximately equal to the strain which can actually be recovered in an experiment on retardation of strain at $P_- = 0$.

It now appears better to define the elastically recoverable strain under conditions in which the whole stress present in the medium is different from zero and corresponds to the stored elastic energy, as in an experiment on the relaxation of stress at $\dot{S}^- = 0$ ($S =$ constant). According to Maxwell's law one can then define an elastically recoverable strain as the ratio of the stress and the elasticity modulus, calculated by dividing the viscosity modulus by the relaxation time observed in the said relaxation experiment $\dot{S}^- = 0$.

Alternatively, one may use in the terminal position vibrational experiments, and extrapolate the experiments to zero frequency and zero amplitude. One there deduces the

elastically recoverable strain by dividing the stress by an elastic modulus calculated from the reciprocal complex modulus.

Having defined the elastically recoverable strain in any of the above-mentioned ways, one can identify it with the corresponding strain which in an ideally elastic medium can be completely recovered. By doing this, one obtains for all media and conditions not only the information available through the formulations 1 and 2 of the principle but also all the new information which can be deduced from the various formulations of the principle given below for ideally elastic media. It may be noted that the said new information is applicable to media of all kinds, even to media which are considered purely viscous (irrespective of whether the viscosity is normal or abnormal, i.e., constant or dependent on the rate of shear), as such media can be treated in accordance with Maxwell's theory as having an elastically recoverable strain of infinitesimally small size corresponding to the strain that had occurred during the infinitesimally small relaxation time of such media.

4. Theory of Elasticity

For the clear case of ideally elastic media we shall now reformulate the principle by considering that on every test area the normal, tangential and cross directions are now uniquely defined, not only for the tractional forces but also for the tractional displacement of the test area relative to its

parallel neighbor, because for elastic media one has to take into account only the total tractional displacement. From assumption IX.2 and equation (1) one then deduces for any one test area:

Principle for Elastic Media (Formulation 3): For every test area of an elastic medium one has a coincidence of the normal, tangential and cross direction of the tractional forces with the corresponding directions of the tractional total displacement of the test area relative to its parallel neighbor, this displacement being taken from an initial position of isotropic symmetry to the terminal position considered. In short, an elastic medium exhibits at every test area stresses and strains which are anisotropically similar in an invariant sense (i.e., independent of the arbitrary choice of the scale functions).[1]

The tensorial equation of state corresponding to this formulation can therefore be written in any or all of the forms given in lecture VIII for the anisotropic similitude. Moreover, some further forms can be derived which all give different aspects of the invariant anisotropic similitude. In particular, it is found that for elastic media s(t) in (1) is no longer a function of time.

1. This follows from the identity of formulation 3 of the principle with the invariant definition of the anisotropic similitude of states of strains and stresses (see lecture VIII).

The corresponding algebraic formulation of the tensorial equation of state is then deduced; s(t) is no longer a function of time but uniquely defined as s say, and therefore,[1]

$$R_G [F, s, n] = R_G [s,n] = \begin{cases} 2 \\ 1 \end{cases} \text{ with } G(F,n,s) = 0 \qquad (6)$$

This means that in anisotropically similar states of stresses and strains the tractional force lies always in the plane containing the normal n to the test area, and the vector s which characterizes the total tractional displacement.

One can express equation (6) in terms of the scale functions S^- and P^- and then in terms of any arbitrarily chosen scale functions C^- and D^-, and then finds[2]

$$R_G = \begin{cases} 2 \\ 1 \end{cases} \text{ for } G(I^-, S^-, P^-) = 0 \text{ with } \qquad (7)$$

$$S^- = c_o I^- + c_1 C^- + c_2 C^- C_- \qquad \text{and} \qquad (8)$$

_____

1. The rank of Gram's determinant can never be larger than the number of vectors in the set, and must therefore be equal to 2 or 1 for the Gram's determinant of s and n.

2. See Grossman, P.U.A., l.c. One can define for symmetric tensors Gram's determinant by writing each term in that determinant as the scalar double dot product of the tensors instead of the scalar dot product of the vectors. The same applies to volume densities and capacities of tensors.

Proceeding then from the consideration of one test area to a comparison of the conditions on two test areas of different orientation, one can now formulate for elastic media the assumption IX.3 and the equation (2) for all three mutually perpendicular components without reference to directions and time dependence. This will reveal the consistency of the assumptions with one another, and in each assumption for the three components involved, as one deduces here the same scale invariant anisotropic similitude of stress and strain from considering separately for the various components assumption IX.3 and equation (2). Incidentally, some further aspects of the anisotropic similitude of stress and strain will thereby be revealed which are again scale invariant but refer instead of the directions to the amounts of the components involved.

Principle for Elastic Media (Formulation 4): One finds on two test areas of different orientations the same amounts of normal (tangential, cross) components of the traction forces per unit area if the two test areas have suffered per unit area the same amounts of the normal (tangential, cross) components of the total fractional displacements relative to their parallel neighbors, the total displacement being taken from an initial position of isotropic symmetry to the terminal position considered.

By formulating then equation (2) separately for each of the three components, and considering that n_- and m_- are no

longer functions of time, but uniquely defined, one obtains
three equations from which one can deduce equations (6) to
(9) and so establish here again the anisotropic similitude,
now formulated in a scale-invariant way by reference to the
amounts of the components.[1]

There is no need to give the relevant equations here
as they are identical with those already given. However, in
dealing with elastic media one has to take into account that
any equation of state formulated for such media will have to
be restricted in accordance with the existence of a strain
potential $\bar{\Phi}$ say, which ensures that the work done on any com-
plete cycle equals zero. One can express this restriction in
terms of the scalar invariants of the scale functions S^-, _viz._,

$$\frac{\partial \bar{\Phi}^-}{\partial S^-_{II}} = 0 \text{ (for } S^-_I = \text{const. and } S^-_{III} = \text{const.)} \quad (10)$$

and in terms of any arbitrary scale function C^- by substituting
in (10) in accordance with (8).

5. Application to Simple (Two-Dimensional) Laminar Shearing Motions

Against the many applications there is one which is
particularly well-suited for a discussion and that is the me-
chanical behavior of media under the application of simple (two-
dimensional) laminar shearing motions. These motions have al-
ready been considered earlier on the basis of the tensorial

---

1. See Grossman, P.U.A., _l.c._

equations of state derived from formulations 1 and 2 of the
principle. The conclusion then was the equality of P_{22} and
P_{33} should apply to all media -- solid, fluid, or intermediate,
elastic, viscous, or viscoelastic, compressible or incompress-
ible, etc., -- and to all the various boundary conditions, and
conditions of flow compatible with a simple (two-dimensional)
laminar shearing motion. Now, the same generality of appli-
cation will be claimed for conclusions reached from the equa-
tions of state derived from formulations 3 and 4 of the prin-
ciple in which the prehistory of the tractional movement is
replaced by an appropriately defined elastically recoverable
strain.

From considerations of symmetry one finds for every
simple (two-dimensional) laminar shear in Cartesian systems of
coordinates that stress P is given by

$$P = \begin{vmatrix} P_{11} & P_{12} & 0 \\ P_{12} & P_{22} & 0 \\ 0 & 0 & P_{33} \end{vmatrix} \qquad (11)*$$

while the elastically recoverable strain is given by

$$S = \begin{vmatrix} c_{+1}^2 & c & 0 \\ c & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad \text{where} \qquad (12)*$$

C is the displacement of a plane at unit normal distance from the

plane of the test area and the strain velocity[1] $\dot{s}$ by

$$\dot{s} = \begin{vmatrix} 0 & \dot{s}_{12} & 0 \\ \dot{s}_{12} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \quad \text{for } \dot{s} \neq 0 \qquad (13)*$$

Denoting by α_{ik}, β_{ik} and γ_{ik} the angles between the k axis of coordinates and the i main axis of the stress P, the elastically recoverable strain S, and the strain velocity $\dot{s}$ respectively, one finds from (11)* to (13)*

$$\alpha_{k3} = \beta_{k3} = \gamma_{k3} = \alpha_{3k} = \beta_{3k} = \gamma_{3k} = 0 \quad \text{for } k = 1,2,3 \qquad (14)*$$

$$\alpha_{11} = \alpha_{22} = -\alpha_{12} = 180° - \alpha_{21} = 1/2 \arctan \frac{2 \, P_{12}}{P_{11} - P_{22}} \qquad (15)*$$

$$\beta_{11} = \beta_{22} = -\beta_{12} = 180° - \beta_{21} = 1/2 \arctan \frac{2}{C} \qquad (16)*$$

$$\gamma_{11} = \gamma_{22} = -\gamma_{12} = 180° - \gamma_{21} = 45° \qquad (17)*$$

The formulations 3 and 4 of the principle then require that there should be an anisotropic similitude between the stress and the elastically recoverable strain so that P and S obey all the equations (6)* to (9)* of this lecture as well as the corresponding equations of lecture VIII. It follows from all these equations that in accordance with the said aniso-tropic similitude there must always be coaxiality between P and S (but not in general with $\dot{s}$), viz.,

$$\alpha_{ik} = \beta_{ik} \quad \text{(for i and k equal to 1,2,3)} \qquad \text{with} \qquad (18)*$$

$$\alpha_{kk} = \beta_{kk} = 1/2 \arctan \frac{2P_{12}}{P_{11} - P_{22}} = 1/2 \arctan \frac{2}{C} \quad \text{(for k=1,2)} \quad (19)*$$

and with equations similar to (19)* for the indices 12 and 21 of α and β. It also follows that there must always be proportionality between the differences of the components of P with cyclically permuted indices and the differences of the corresponding components of S, with a scalar proportionality factor γ, say, viz.,

$$P_{11} - P_{22} = \quad ([c^2+1]-1) = \quad c^2 \tag{20}*$$

$$P_{22} - P_{33} = \quad (1-1) = 0 \tag{21}*$$

$$P_{33} - P_{11} = \quad (1-[c^2+1]) = - \quad c^2 \tag{22}*$$

$$P_{12} - 0 = \quad (c-0) = \quad c \tag{23}*$$

where the scalar proportionality factor γ may be regarded as a shear modulus of elasticity of the kind of a pseudoproperty of the medium as γ will, in general, not be a constant but a function of any or all the scalar parameters, including the scalar invariants of the stress, of the elastically recoverable strain, and of the strain velocity, etc. One finds for the said shear modulus from (20)* to (23)*

$$\gamma = \frac{P_{12}^{\ 2}}{P_{11}-P_{22}} \qquad \text{and} \tag{24}*$$

for the elastically recoverable strain a Cartesian tensor of the form (12)* with

$$c = \frac{P_{11}-P_{22}}{P_{12}} \tag{25}*$$

(Here then one has a special case in which the elastically recoverable strain is already determined by the principle, and

one can check by experiments on relaxation, retardation, and
on superposition of vibrations, which general definition of
elastically recoverable strain will best be used as an equiva-
lent for the generally unknown prehistory of the medium.)

Summarizing the results obtained above for the orien-
tations of the Cartesian tensors P, S and $\dot{S}$,[1] one finds in the
coordinate plane (12)

$$\alpha_{kk} = \beta_{kk} = 45° - \psi \quad \text{(for k=1,2)} \quad \text{with} \quad (26)*$$

$$\psi = 1/2 \arctan \frac{P_{11} - P_{22}}{2P_{12}} = 1/2 \arctan \frac{c}{2} \quad (27)*$$

(a) The differences between the traction forces
across test planes parallel and perpendicular to the tangential
direction of the shearing movement must have the signs of pulls,
and the amounts of these pulls per unit area should correspond
to the amounts of the tangential components of the elastically
recoverable strain. Hence, in combination with (5)* one finds

$$P_{11} - P_{22} = P_{11} - P_{33} = \gamma c^2 \geqslant 0 \quad (28)*$$

when pulls are associated with a positive sign.

(b) The differences between the amounts of the trac-
tional forces per unit area parallel and perpendicular to the
tangential directions of the shearing movement will have amounts
which are infinitesimally small, approaching zero in the limit.

1. It suffices to discuss the orientations of the tensors in the
(12) coordinate plane since there is a coincidence of the No. 3
main axes of all the said tensors with the No. 3 coordinate axes.

This means that whenever the amount of the shear component C of the elastically recoverable strain is finite, or vanishingly small, one will observe respectively finite amounts, or vanishingly small amounts of normal pressure effects, i.e., of differences $(P_{11} - P_{22})$ of normal pressures, and one will also observe finite amounts, or vanishingly small amounts of the angle by which the directions of the main axes No. 1 and 2 of the stress P and of the recoverable strain S deviate from the $\pm 45^{\circ}$ directions in the (12) coordinate plane. These observations are expected in media of all consistencies and of all mechanical properties -- solid, fluid or intermediate. It may be noted that Newtonian media and all other media which are described as purely viscous will here be regarded as limiting cases of viscoelastic media with vanishingly small (or zero) amount of the shear component C of the elastically recoverable strain. It then follows that all these media will not show any normal pressure effects (the difference $P_{11} - P_{22}$) being then vanishingly small, as well as the differences $(P_{22} - P_{33})$ and $(P_{33} - P_{11})$), and there will be no deviation of the main axes No. 1 and 2 of the stress and the recoverable strain from the $\pm 45^{\circ}$ directions in the (12) coordinate plane. As the $\pm 45^{\circ}$ directions coincide with the directions of the main axes No. 1 and No. 2 of the strain velocity $\dot{S}$ for all media, and all rates of shear different from zero, one finds

here that there is coaxiality of stress, recoverable strain
and strain velocity.

The main result has been the prediction of phenomena
of normal pressures in simple (two-dimensional) laminar shearing
motions. The prediction was that the differences between the
normal pressures per unit area taken on two test areas respec-
tively perpendicular and parallel to the lines of flow should
have the sign of pulls and correspond to the elastically re-
coverable amounts of normal displacements of these areas rela-
tive to their parallel neighbors. These pulls are expected
to approximate zero for all media and conditions under which
the recoverable elastic strain approaches zero, as for instance
for all purely viscous media (including Newtonian fluids) under
all conditions, and for viscoelastic media under vibrational
tests with small amplitudes approximating zero. It was fur-
ther predicted that there should be an angular deviation be-
tween the directions of the main axes of stress, and those of
the strain velocity, this deviation corresponding to the amount
of elastically recoverable strain and approximating zero if
and when the amount of elastically recoverable strain approxi-
mates zero.

Seminar on Continuum Mechanics

Lecture X

Principles of Mechanical Properties of Continuous Media

Part V.   The Tensorial Equation of State Deduced by Generalization

A.   Introduction

   1.   Outline of General Procedure

B.   Generalization of Hooke's Law of Elasticity by Means of
       Scale Invariants

   1.   Formulation of Hooke's Law in Cartesian Systems of
          Coordinates

   2.   The Anisotropic Similitude of Stresses and Infinitesi-
          mally Small Strains Deduced from the Scale-Invari-
          ant Formulation of Hooke's Law

   3.   Scale-Invariant Generalization of Hooke's Law Result-
          ing in a Law of Anisotropic Similitude between
          Stresses and Strains of any Amount, Large or Small

C.   Generalization of Time-Dependent Linear Laws for Infinitesi-
       mally Small Strains

D.   Discussion of Usefulness of the Principle

Seminar on Continuum Mechanics

Lecture X

Principles of Mechanical Properties of Continuous Media

Part V. The Tensorial Equation of State Deduced by Generalization

A. Introduction

  1. Outline of General Procedure

    The tensorial equation of state will be derived in
this lecture in a manner distinctly different from that adopted in
the previous lecture which had been based on an analysis of the
tractional movement and tractional forces into their respective
normal, tangential and cross components.

    The procedure adopted in this lecture starts from the
consideration of a special case -- that of an ideally elastic
medium subjected to strains of infinitesimally small amounts,
for which a tensorial equation of state has already been es-
tablished as Hooke's law of elasticity -- and proceeds from
there to build up in successive steps of generalization the
principle of a tensorial equation of state which should be
applicable to all media and conditions. Each step of this gen-
eralization is based on the use of scale invariants, and this
makes the generalization uniquely determined, as there is one
and only one way in which the correlation of Hooke's law be-
tween the stresses and the infinitesimally small strains can

be made invariant against all changes of the scale functions
by which the stresses and strains are measured.

After the generalization of Hooke's law has been
completed as indicated above, an analogous procedure will be
adopted for the generalization of time-dependent linear laws
which have been established for infinitesimally small strains.

It will be seen that in this way the formulations of
the principle and of the equation of state given in the pre-
vious lecture will be regained, so that the self-consistency
of the whole scheme will be demonstrated.

For ease of notation the equations will be restricted
to Cartesian systems of coordinates except where explicitly
stated otherwise. The said restriction (denoted again by an
asteric attached to the number of the equation) will allow us
to disregard the distinction between true tensors and pseudo-
tensors (volume densities and capacities of tensors) and to
treat all these quantities as true Cartesian tensors and to
denote them all by symbols without the distinguishing stroke
at index level which has been used for the pseudo-tensors.
Thus, the stress, which is a tensor density, will be denoted
by P in equations with an asteric and by P^- in equations with-
out an asteric, as the latter apply to all coordinate systems,
Cartesian as well as non-Cartesian.

It will be convenient to retain the assumption
IX.1 in the present lecture and to postulate that the media
have in the initial position the full symmetry of the Euclid-
ean space in which they are imbedded, i.e., they are in that
position (but not in general in any other position) homogene-
ous, isotropic and parity-symmetrical under mechanical testing.

B. Generalization of Hooke's Law of Elasticity by Means of

Scale Invariants

1. Formulation of Hooke's Law in Cartesian Systems of

Coordinates

One can write Hooke's law in Cartesian system of
coordinates as

$$P + \beta\epsilon + \gamma\epsilon_I = 0 \qquad\qquad (1)*$$

where one denotes by

P   the stress

ϵ the strain of an infinitesimally small amount which

according to the classical definition is given

as the symmetrical part of the gradient of the

displacement vector

ϵ_I the first invariant of ϵ

β, γ material constants.

The formulation of Hooke's law in equation (1) is
given in terms of the classical scale functions P and ϵ, but

this formulation is not invariant against changes of these functions. For the infinitesimally small strains one finds that the choice of a scale function is irrelevant since for strains of this size all the scale functions are linearly interrelated, and a change from one to another will only affect the values of the material constants but not their constancy nor the linearity of the equation (1). However, the change of scale functions will be relevant for the stresses, and also for strains whose sizes are unrestricted so that they may be of any amount, large or small. A change in the scale functions will then in general affect not only the values but also the constancy material characteristics, and will moreover change the linearity of the equation (1) making it only tensor-linear or nonlinear altogether. As the choice of scale functions is arbitrary, a generalization of Hooke's law which is dependent on some particular choice of these functions will involve the same arbitrariness. In order to avoid such an arbitrariness one has to find a formulation of Hooke's law which is invariant against all changes of the scale functions, and this formulation can then be generalized under preservation of the said invariance.

2. The Anisotropic Similitude of Stresses and Infinitesimally Small Strains Deduced from the Scale-Invariant Formulation of Hooke's Law

The Scale-invariant formulation of Hooke's law can be obtained in two distinct stages, _viz_.

In the first stage we eliminate the material constants. For this purpose we apply to the Cartesian coordinate system certain rotations which will result in successive cyclical permutations of the coordinate axes, thus yielding in the original Cartesian system the equations

$$P' + \beta\epsilon' + \gamma\epsilon_I I = 0 \qquad \text{and} \qquad (2)$$

$$P'' + \beta\epsilon'' + \gamma\epsilon_I I = 0 \qquad (3)$$

where the stresses and strains obtained by the successive cyclical permutations are denoted by P' and ϵ' and by P'' and ϵ'' respectively. The quantities β, γ, ϵ_I and I have remained unchanged because they are by definition of isotropic symmetry, and hence invariant against the said permutations. Subtracting then in cyclical order equation (2) from (1), (3) from (2) and (1) from (3), one finds for the ratio of the differences of the stresses and strains

$$(P-P'):(P'-P''):(P''-P) = (\epsilon-\epsilon'):(\epsilon'-\epsilon''):(\epsilon''-\epsilon') \qquad (4)*$$

and this equation is invariant against all linear changes of the scale functions P and ϵ but not against nonlinear changes.

In the second step we compare the equation (4)* with the equation (34)* of the lecture VIII, which expressed the anisotropic similitude, and find that both equations are identical for the strains of infinitesimally small amounts here considered.[1] As the anisotropic similitude had been defined in a way completely invariant against changes in the scale functions by the coincidence on every test area of the normal, tangential and cross directions of the tractional forces with the corresponding directions of the tractional movements there. One can easily verify[2] that Hooke's law implies this invariantly defined anisotropic similitude by equation (4)*.

---

1. Equation (34)* of lecture VIII can be written as

$$(P-P'):(P'-P''):(P''-P) = (S-S'):(S'-S''):(S''-S)$$

and this is further equal to

$$[(\lambda_1\lambda_2\lambda_3 S - I)-(\lambda_1\lambda_2\lambda_3 S'-I)]:[(\lambda_1\lambda_2\lambda_3 S'-I)-(\lambda_1\lambda_2\lambda_3 S''-I)]:[(\lambda_1\lambda_2\lambda_3 S''-I)-(\lambda_1\lambda_2\lambda_3 S-I)]$$

which in the limes of infinitesimally small strains (i.e.,

$\lim \lambda_i = 1$) becomes equation (4) since

$$\epsilon = 1/2 \lim_{\lambda_i \to 1}(\lambda_1\lambda_2\lambda_3 S-I) \qquad \underline{q.e.d.}$$

2. For this verification one calculates from (4) and from (15) to (28) of lecture VIII the directions in questions, and finds the coincidence of the directions of corresponding components by direct comparison of the results. Alternatively, one can compare (4)* with (31) of lecture VIII, and show that (4)* is a special case of (31) for the limit of infinitesimally small strains.

3. Scale-Invariang Generalization of Hooke's Law Resulting in a Law of Anisotropic Similitude between Stresses and Strains of any Amounts, Large or Small

A scale-invariant generalization of Hooke's law can now be carried out by postulating that the invariantly defined anisotropic similitude should be maintained for finite amounts of strain. In that way equation (4)* is then generalized to

$$UP + VS + WI = 0 \qquad\qquad (5)*$$

and this can be written in a manner completely invariant against all coordinate transformations and all changes of scale functions as

$$UP^- + VS^- + WI^- = 0 \qquad \text{where} \qquad (6)$$

$$P^- = d_o I^- + d_1 D^- + d_2 D^-_1 D_- \quad \text{---} \quad \text{and} \qquad (7)$$

$$S^- = c_o I^- + c_1 C^- + c_2 C^- C_- \qquad\qquad (8)$$

and where U, V and W are scalars which can now be either constants or functions of any scalar parameters including the scalar invariants of P^- and S^-. The C^-_n and D^-_n are arbitrary scale functions, and their introduction into (6) gives an equation identical with (4) of lecture VIII. Thus, the law derived here by generalization of Hooke's law is a tensorial equation of state identical with that deduced in the previous lecture for ideally elastic media by the analysis of the tractional forces and tractional movements into the three mutually

perpendicular components. The equations (5)* to (8) can then
be rewritten in any or all the forms given in lectures VIII
and IX for the anisotropic similitude of stresses and strains.
However, one has to remember that the stress strain relations
for ideally elastic media are restricted by the assumption
of the evidence of a strain potential which ensures that the
external work is zero for any complete cycle of straining move-
ments. This condition is automatically fulfilled for media
which obey Hooke's law, but is not in general fulfilled for
media obeying the generalized law. For such media one has
to add the condition for a strain potential, as given in the
previous lecture if one wants to restrict the generalized law
to ideally elastic media.

C. Generalization of Time-Dependent Linear Laws for Infini-
tesimally Small Strains

Within the frame work of classical mechanics of in-
finitesimally small strains, one accounted for the effects of
the prehistory by formulating a law in which one related a
linear aggregate of the stresses and their time derivatives, $\overset{(i)}{\rho}$
say, to a linear aggregate of the strains and their time deriva-
tives $\overset{(j)}{\epsilon}$ say, the strains being infinitesimally small, and the
linear aggregates having constant coefficients $\pi_{(i)}$, $\sigma_{(j)}$
and $\tau_{(k)}$ say, viz., in notation invariant to transforma-
tions of Cartesian coordinate systems

$$i\sum_0^{\infty} \pi_{(i)} \overset{(i)}{P} = j\sum_0^{\infty} \sigma_{(j)} \overset{(j)}{\epsilon} + k\sum_0^{\infty} \tau_{(k)} \overset{(k)}{\epsilon_I} I \qquad (9)*$$

where, for ease of description, one denoted the quantities

P, ϵ and ϵ_I respectively by $\overset{(i)}{P}$, $\overset{(j)}{\epsilon}$, and $\overset{(k)}{\epsilon}_I$ for i, j and

k = 0. All the linear laws in classical mechanics of infini-

tesimally small strains are special cases of (9)*; in particu-

lar one finds for solid media that Hooke's law of elasticity

and Kelvin's law of viscoelasticity are such special cases,

and so are the laws for fluid media known as Newton's law

of viscosity and Maxwell's law of viscoelasticity.

For a generalization one again eliminates the con-

stant coefficient. This can be done in two stages. In the

first stage we apply again rotations corresponding to cycli-

cal permutations of the coordinate axes, and eliminate the

constants $\tau_{(k)}$ by subtracting the equations from one

another. One then obtains three equations, viz.,

$$i\sum_0^{\infty} \pi_{(i)} (\overset{(i)}{P} - \overset{(i)}{P'}) = j\sum_0^{\infty} \sigma_{(j)} (\overset{(j)}{\epsilon} - \overset{(j)}{\epsilon'}) \qquad (10)*$$

and two further equations of the same type for $(\overset{(i)}{P'} - \overset{(i)}{P''})$ and

$(\overset{(i)}{P''} - \overset{(i)}{P})$. In the second we assume a special prehistory with

a periodic straining movement. For this prehistory one finds

from (9) periodic solutions for the stresses and strains, for

which one has in the usual complex notation

$$(P-P'):(P'-P''):(P''-P) = (\epsilon-\epsilon'):(\epsilon'-\epsilon''):(\epsilon''-\epsilon') \qquad (11)*$$

where all the quantities in (11) are complex. This means that

in case of such a special periodic prehistory one finds again

anisotropic similitude for the stresses and infinitesimally

small strains, and this similitude can easily be verified

first for the special prehistory and the infinitesimally

small strains considered as there will be on a test area of any

orientation a coincidence of the normal tangential and cross

directions of the traction forces with the corresponding direc-

tions of the displacement of that plane relative to its par-

allel neighbor. There will also be on any two test areas of

different orientations the same normal (or tangential or

cross) components of the traction forces per unit area, for

the same corresponding components per unit area of the dis-

placements of the areas with respect to their parallel neighbors.

For the said special prehistory and infinitesimally

small strains one can then generalize the law (9) which is

truly linear with constant coefficients, and obtain a gener-

alized law which is only tensorial linear (i.e., involving

only the first powers of tensors) but scalar nonlinear, by

replacing the constant coefficients by functions of any scalar

variables (including the three scalar invariants of the stresses

and strains). This law will still postulate the anisotropic

similitude for the stresses and strains under the conditions

already specified, as one can derive from it an equation iden-

tical with (11) by proceeding as above.

A further generalization can be made by taking into account prehistories of any kind, and strains of any magnitude. In the course of such a prehistory with strains of finite amounts the tangential directions of the displacement of a test area with respect to its parallel neighbors will in general not be uniquely defined but cover a whole range of directions, and only this range, but not a single member of it, can be related to the tangential direction of the traction force present at any instant of time. Thus, the postulate of an anisotropic similitude which is invariant of any changes of the scale functions and based on coincidence of directions cannot be maintained in this generalization. For the same reason one cannot relate the equality of tangential components of the traction forces on two test areas with the equality of the corresponding components in the displacements of the two test areas with respect to their parallel neighbors. However, this difficulty does not exist for the normal directions of two test areas as these directions will always be uniquely defined, and will be the same for the traction forces and the relative displacements. Bearing this in mind one finds that the required generalization can be carried out in a manner which is invariant against changes in the scale functions by adopting the principle as given in its first and second formulation in the preceding lecture.

Thus, altogether one arrives by a synthetic process of successive steps of generalization at the same formulations of the principle and the tensorial equations of state as had previously been deduced by the analytical procedure. This completes the cycle and emphasizes the self-consistency of the whole scheme.

D. Discussion of the Usefulness of the Principle

In a study of Continuum Mechanics the principle can be usefully applied in many ways as it serves as a guide to the phenomena in which there is a likeness in the mechanical behavior of continuous media of all kinds and under all conditions.

With regard to theoretical applications it is interesting to note that the exact validity of the principle can be established for certain groups of ideal media, and for certain conditions of symmetry, viz.,

(1) for the groups of ideal media which are defined as obeying a linear relationship between the stresses and infinitesimally small strains and their respective time derivatives. The principle is valid within the range of such small strains, irrespective of whether the media are solids, obeying Hooke's law of elasticity or Kelvin's law of viscoelasticity, or fluids obeying Newton's law of viscosity, or Maxwell's law

of viscoelasticity or any other linear law. (This follows from the consideration given in chapter D of the present lecture.)

(2) for deformations of cylindrical or higher symmetry, when they are applied in any amounts, large or small, and in any development of time to a medium of any kind and consistency. (To prove this one can derive the formulation 3 of the principle from the cylindrical or higher symmetry of the applied deformations.)

For a completely general case one cannot prove theoretically the exact validity of the principle. However, it can serve there for the prediction of phenomena, at least in a first approximation, and one can then carry out experimental tests, and find out what corrections, if any, have to be made in accordance with the terms which have been deliberately neglected in the said first order of approximation. For simple (two-dimensional) laminar shearing motions, predictions of normal pressure phenomena have in fact been made for media of all consistencies -- solid, fluid, and intermediate -- and the experimental tests have led to the following conclusions:

(1) There are very wide ranges of media and conditions for which there is an exact agreement within the limits of experimental accuracy between the predictions of the principle and the experimental results.

(2) For media and conditions outside the ranges
mentioned under (1), deviations were found for some media
and conditions, but these deviations were still sufficiently
small to regard the predictions by the principle as a first
approximation to the experimental results, so much so that
it appears still doubtful whether the deviations were due
to experimental defects, or to the idealization implied by
the principle.

BIBLIOGRAPHY

1. Weissenberg, K. Mech. Def. Koerper, Mitt. d. Staatl. Mat. Pruef. Amts, Sonderheft V, 1929.

2a. Eisenschitz, R., Raboniowitsch, B. and Weissenberg, K. Viscositaet und Elasticitaet von Gelen und Solen, Mitt. d. Staatl. Mat. Pruef. Amts, Sonderheft IX, 1929.

2b. Rabinowitsch, B. Viscositaet und Elasticitaet von Gelen und Solen, Zs. phys. chem. A., pp. 145 ff., 1929; Zs. phys. chem. A., 166, 1933.

3a. Weissenberg, K. Naturwissenschaften, Schwingungs Viscosimeter, 1930.

3b. Eisenschitz, R. and Philipoff, W. Naturwissenschaften, 1931.

4. Weissenberg, K. Mech. Def. Koerper, Preuss. Akad. d. Wiss., Sonderheft 2, Berlin, 1931.

5. _____. Mech. des Corps Deform., Arch. d. Scien. et Nat., Vol. 5, pp. 17 ff., 1935.

6. _____, et. al. Rheol. Leaflet No. 10, pp. 10-20, 1939.

7a. _____. Geometry of Rheol. Phen., Rep. on Conf. Brit. Rheol. Club, London, 1946; Th. Nelson and Sons, Ltd., 1947.

7b. Russel, R. T. Ph.D. Thesis Imp. Coll., London, 1946.

7c. Lodge, A. S. The Comp. Cond. for Large Strains, Qu. J. Mech. and Appl. Math. IV, pp. 85 ff, 1951.

8. Weissenberg, K. Cont. Theory of Rheol. Phen., Nature 159, pp. 310 ff., 1947.

9. _____. The Use of a Trellis Model in Mech. etc., Part I, Shirley Mem. XXI, pp. 203 ff., 1947.

10. _____. Chadwick, G. E., Shorter, S. A. and Weissenberg, I The Use of a Trellis Model in Mech. etc., Part II, Shirley Mem. XXI, pp. 225 ff. 1947.

11. Freeman, S. M. and Weissenberg, K. Some New Rheol. Phen. etc., Nature, Vol. 162, pp. 320 ff., 1948.

12. Freeman, S. M. and Weissenberg, K. Rheol. and Const. of Matter, Int. Cong. on Rheol., Holland, 1948; North Holl. Publ. Co., 1949.

13a. Weissenberg, K. Spec. of Rheol. Phen. by a Rheogoniometer, Int. Cong. on Rheol., Holland, 1948; North Holl. Publ. Co., 1949.

13b. Roberts, J. Press. Distr. in Liquids etc., Min. of Supply, ADE Report 13, 1952.

13c. _____. Press. Distr. in Liquids etc., Proc. Int. Cong. on Rheol., Oxford, pp. 91 ff., 1953.

13d. _____. Normal Stress Eff. etc., Nature, Vol. 179, pp. 487 ff., 1957.

13e. Jobling, A. Ph.D. Thesis, Cambridge University, 1956.

13f. Jobling, A. and Roberts, J. Rheol., Theory and Appl., Vol. II, pp. 503 ff., Academic Press, 1957.

13g. _____. An Investigation with the Weissenberg Rheogoniometer, J. Pol. Sc. XXXVI, p. 433, 1959.

13h. _____. Flow Testing of Visco-Elastic Material, J. Pol. Sc. XXXVI, p. 421, 1959.

13i. Lammiman, K. A. and Roberts, J. E. Meas. of Visco-Elast. etc., Rep. on Rheol. Conf. Llandudno, 1961.

14a. Weissenberg, K. Abn. Subst. and Abn. Phen. of Flow, 2nd Proc. Int. Cong. on Rheol., Holland, 1948; North Holl. Publ. Co., 1949.

14b. Grossman, G. U. P. Ph.D. Thesis, Cambridge University, 1948; and "Weissengerg's Equation of Rheol. State," Koll. Zs. 174, 97, 1961.

15. Weissenberg, K. Rheol. of Hydro Carbon Gels, Proc. Royal Society, 200A, pp. 183 ff., 1950.

16a. _____. Inst. for Biaxial Strains, Bull. Br. Socy. of Rheol., No. 43, p. 6, 1955.

16b. Grossman, G. U. P. A Simple Inst. etc., J. Science Inst.
        35, 131, 1958; Ph.D. Thesis, Cambridge University,
        1958.

17. Weissenberg, K. The Rheogoniometer, Proc. of Oxford Meeting
        of Faraday Society and British Society of Rehology,
        1959; Pergamon Press, 1960.

18. _____. "Some Demonstrations of the Application of
        Rheology etc."(Flow Properties of Blood, Pergamon
        Press, 1960).

DISTRIBUTION LIST

for

Contract Nonr 266(08)

Administrative and
Liaison Activities

Chief of Naval Research
Department of the Navy
Washington 25, D. C.
Attn: Code 418   (1)
        Code 438   (2)

Office of Naval Research
c/o Hudson Laboratory
148 Palisades Avenue
Dobbs Ferry, New York   (1)

Commanding Officer
Office of Naval Research
Branch Office
John Crerar Library Building
86 E. Randolph Street
Chicago 11, Illinois   (1)

Commanding Officer
Office of Naval Research
Branch Office
346 Broadway
New York 13, N. Y.   (1)

Commanding Officer
Office of Naval Research
Branch Office
1030 E. Green Street
Pasadena, California   (1)

Commanding Officer
Office of Naval Research
Navy 100, % Fleet Post Office
New York, N. Y.   (2)

Director
Naval Research Laboratory
Washington 25, D. C.
Attn: Tech. Info. Officer   (6)
        Code 6200   (1)
        Code 6205   (1)
        Code 6250   (1)
        Code 6260   (1)
        Code 2029   (1)

Armed Services Technical Information Agency
Arlington Hall Station
Arlington 12, Virginia   (10)

Office of Technical Services
Department of Commerce
Washington 25, D. C.   (1)

U. S. Department of Defense

Office of the Secretary of Defense
Research and Development Division
The Pentagon
Washington 25, D. C.
Attn: Technical Library   (1)

Chief, Defense Atomic Support Agency
The Pentagon
Washington 25, D. C.
Attn: Document Library Branch   (2)

Commanding General
Field Command
AFSWP, P. O. Box 5100
Albuquerque, New Mexico   (3)

Army

Office of the Secretary of the Army
The Pentagon
Washington 25, D. C.
Attn: Army Library   (1)

Chief of Staff
Department of the Army
Washington 25, D. C.
Attn: Develop. Branch (R and D Div.)   (1)
       Research Branch (R and D Div.)   (1)
       Special Weapons Br.(R and D Div.)(1)

Office of the Chief of Engineers
Asst. Chief of Military Construction
Department of the Army
Bldg. T-7, Gravelly Point
Attn: Library Branch   (1)
       Structural Branch (Engr. Div.)   (1)
       Protective Construction Branch
       (Pl., Engr., and Contracts)   (1)

ARMY (Continued)

Chief, Research and Devel.   h
Department of the Army
Washington 25, D. C.
Attn: Special Weapons and
       Air Defense Div.   (1)

Chief of Engineers
Department of the Army
Washington 25, D. C.
Attn: ENGNB   (1)

Commanding General
Aberdeen Proving Ground
Maryland
Attn: Director, Ballistic
       Research Lab.   (2)

Commanding Officer, Engineer
Research and Development Lab.
Fort Belvoir, Virginia
Attn: Chief, Tech. Intelligence
                    Br.   (1)

Director, Waterways Experiment
            Station
P. O. Box 631
Vicksburg, Miss.
Attn: Library   (1)

Office of the Chief of Ordnance
Department of the Army
Washington 25, D. C.
Attn: Research and Materials Br.
       (Ord. R and D Div.)   (1)
       ORDTX - AR   (1)

Office of the Chief Signal Officer
Department of the Army
Washington 25, D. C.
Attn: Engin. and Tech. Div. (1)

Commanding Officer
Watertown Arsenal
Watertown, Massachusetts
Attn: Laboratory Div.   (1)

Commanding Officer
Frankford Arsenal
Bridesburg Station
Philadelphia 37, Pa.
Attn: Laboratory Div.   (1)

Office of Ordnance Research
2127 Myrtle Drive
Duke Station
Durham, North Carolina
Attn: Div. of Engin. Sciences(1)

Commanding Officer
Engin. Research Develop. Lab.
Fort Belvoir, Virginia   (1)

Chief of Naval Operations
Department of the Navy
Washington 25, D. C.
Attn: Op 37   (1)
       Op 36   (1)
       Op 03EG   (1)

Commandant, Marine Corps
Headquarters, U. S. Marine Corps
Washington 25, D. C.   (1)

Chief, Bureau of Ships
Department of the Navy
Washington 25, D. C.
Attn: Code 327   (2)
       Code 348   (1)
       Code 371   (1)
       Code 420   (2)
       Code 423   (2)
       Code 442   (2)
       Code 421   (1)

Chief, Bureau of Aeronautics
Department of the Navy
Washington 25, D. C.
Attn: AD-2   (1)
       AD-22   (1)
       RS-7   (1)
       TD-42   (1)

Chief, Bureau of Ordnance
Department of the Navy
Washington 25, D. C.
Attn: Ad3   (1)
       Re   (1)
       Re3   (1)
       Re5   (1)
       ReN   (1)

Director of Naval Intelligence
Navy Department
Washington 25, D. C.
Attn: Op-922V   (1)

Chief, Bureau of Yards and Docks
Department of the Navy
Washington 25, D. C.
Attn: Code D-213   (1)
       Code D-222   (1)
       Code D-410C   (1)
       Code D-440   (1)
       Code D-500   (1)

NAVY (Continued)

Commanding Officer and Director
David Taylor Model Basin
Washington 7, D. C.
Attn: Code 140   (1)
        Code 600   (1)
        Code 700   (1)
        Code 720   (1)
        Code 725   (1)
        Code 731   (1)
        Code 740   (2).

Commander
U. S. Naval Ordnance Laboratory
White Oak, Maryland
Attn: Tech. Library   (2)
        Tech. Evaluation Dep. (1)
        EE   (1)
        EH   (1)
        R   (1)

Director
Material Laboratory
New York Naval Shipyard
Brooklyn 1, New York   (1)

Commanding Officer and Director
U. S. Naval Electronics Lab.
San Diego 52, California
Attn: Code 4223   (1)

Officer-in-Charge
Naval Civil Engin. Research
Naval Civil Engin. Research and
     Evaluation Laboratory
U. S. Naval Construction
     Battalion Center
Port Hueneme, California
Attn: Code 753   (1)

Director
Naval Air Experimental Station
Naval Air Material Center
Naval Base
Philadelphia 12, Pa
Attn: Materials Laboratory (1)
        Structures Lab.   (1)

Officer-in-Charge
Underwater Explosion Research Div.
Norfolk Naval Shipyard
Portsmouth, Virginai
Attn: Dr. A.H. Keil   (2)

Commander
U.S. Naval Providing Grounds
Dahlgren, Virginia   (1)

Superintendent
Naval Gun Factory
Washington 25, D. C.   (1)

Commander
Naval Ordnance Test Station
Inyokern, China Lake, California
Attn: Physics Division   (1)
       Mechanics Branch   (1)

Commander
Naval Ordnance Test Station
Underwater Ordnance Division
3202 E. Foothill Boulevard
Pasadena 8, California
Attn: Structures Division   (1)

Commanding Officer and Director
Naval Engineering Experiment Station
Annapolis, Maryland   (1)

Superintendent
Naval Post Graduate School
Monterey, California   (1)

Commandant
Marine Corps Schools
Quantico, Virginia
Attn: Director, Marine Corps
       Development Center   (1)

AIR FORCE

Commanding General
U. S. Air Force
Washington 25, D. C.
Attn: Research and Development Div.(1)

Commander
Air Materiel Command
Wright-Patterson Air Force Base
Dayton, Ohio
Attn: WCOSI   (1)

Commander
U. S. Air Force Institute of Technology
Wright-Patterson Air Force Base
Dayton, Ohio
Attn: Chief, Applied Mech. Group   (1)

Director of Intelligence
Headquarters, U. S. Air Force
Washington 25, D. C.
Attn: P. V. Br. (Air Targets Div.) (1)
       AFOIN-1B2   (2)

Commander
Air Research and Development
    Command
P. O. Box 1395
Attn: RDMPE   (1)

Commander
WADD
Wright-Patterson Air Force Base
Dayton, Ohio
Attn: WWRC   (1)
      WWRMDS   (1)
      WWRMDD   (1)

Commanding Officer
USNNOEU
Kirtland Air Force Base
Albuquerque, New Mexico
Attn: Code 20
      (Dr. J. N. Brennan)   (1)

OTHER GOVERNMENT ACTIVITIES

U. S. Atomic Energy Commission
Washington 25, D. C.
Attn: Director of Research (2)

Director
National Bureau of Standards
Washington 25, D. C.
Attn: Division of Mechanics (1)
      Engin. Mechanics Sec. (1)
      Aircraft Structures (1)

Commander
U. S. Coast Guard
1300 E. Street, N. W.
Washington 25, D. C.
Attn: Chief, Testing and
      Development Div.   (1)

U.S. Maritime Administration
General Administration Office Bldg.
Washington 25, D. C.
Attn: Chief, Div. of Preliminary
      Design   (1)

National Advisory Committee for
      Aeronautics
1512 H. Street, N.W.
Washington 25, D. C.
Attn: Loads and Structures Div.
                    (2)

Director
Langley Aeronautical Laboratory
Langley Field, Virginia
Attn: Structures Div.   (2)

Director
Forest Products Laboratory
Madison, Wisconsin   (1)

Civil Aeronautical Administration
Department of Commerce
Washington 25, D. C.
Attn: Chief, Airframes and Equip.Br.(1)

National Sciences Foundation
1520 H. Street, N. W.
Washington 25, D. C.
Attn: Engin. Science Division   (1)

National Academy of Science
2101 Constitution Avenue
Washington 25, D. C.
Attn: Tech. Director, Committee On
      Ships' Structural Design   (1)
      Executive Secretary, Committee
      on Underwater Warfare   (1)

Director, Operations Research Office
John Hopkins University
7100 Connecticut Avenue
Chevy Chase, Maryland
Washington 15, D. C.   (1)

Dr. Alvin C. Graves, Director
J-Division, Los Alamos Scientific Lab.
P. O. Box 1663
Los Alamos, New Mexico   (1)

U. S. Atomic Energy Commision
Classified Technical Library
1901 Constitution Avenue, N. W.
Washington, D. C.
Attn: Mrs. Jean M. O'Leary
    for Dr. Paul C. Fine   (1)

U.S. Atomic Energy Commission
Classified Tech. Library
Tech. Information Service
1901 Constitution Avenue, N.W.
Attn: Mrs. Jean M.O'Leary   (1)

Sandia Corporation, Sandia Base
Albuquerque, New Mexico
Attn: Dr. Walter A. MacNair (1)

Legislative Reference Service
Library of Congress
Washington 25, D. C.
Attn: Dr. E. Wenk   (1)

U.S. Atomic Energy Commission
1901 Constitution Avenue, N.W.
Washington, D. C.   (1)

OTHER GOVERNMENT ACTIVITIES (Continued)

Massachusetts Institute of
    Technology
Cambridge 39, Massachusetts
Attn: Dr. Charles H. Norris   (1)

INVESTIGATORS ACTIVELY ENGAGED
IN RELATED RESEARCH

Professor Lynn S. Beedle
Fritz Engineering Laboratory
Lehigh University
Bethlehem, Pennsylvania   (1)

Professor H. H. Bleich
Department of Civil Engineering
Columbia University
New York 27, New York   (1)

Professor R. L. Bisplinghoff
Dept. of Aeronautical Engineering
Massachusetts Institute of Techno.
Cambridge 39, Massachusetts   (1)

Professor B. A. Boley
Department of Civil Engineering
Columbia University
New York 27, N.Y.

Professor Eugene J. Brunelle, Jr.
Department of Aeronautical Engin.
Princeton University
Princeton, New Jersey   (1)

Professor G.F. Carrier
Pierce Hall
Harvard University
Cambridge 38, Massachusetts   (1)

Professor J.E. Cermak
Department of Civil Engin.
Colorado State University
Fort Collins, Colorado   (1)

Professor Herbert Deresiewicz
Department of Civil Engin.
Columbia University
632 West 125th Street
New York 27, N.Y.   (1)

Professor Lloyd Donnell
Department of Mechanics
Illinois Institute of Technology
Technology Center
Chicago 16, Illinois   (1)

Professor D. C. Drucker, Chairman
Division of Engineering
Brown University
Providence 12, Rhode Island   (1)

Professor A. C. Eringen
Department of Aeronautical Engineering
Purdue University
Lafayette, Indiana   (1)

Professor W. Flugge
Department of Mechanical Engineering
Stanford University
Stanford, California   (1)

Mr. Martin Goland
Midwest Research Institute
4049 Pennsylvania Avenue
Kansas City 2, Missouri   (1)

Professor J. N. Goodier
Department of Mechanical Engineering
Stanford University
Stanford, California   (1)

Professor L.E. Goodman
Engineer Experiment Station
University of Minnesota
Minneapolis, Minnesota   (1)

Professor W.J. Hall
Department of Civil Engineering
University of Illinois
Urbana, Illinois   (1)

Professor R.P. Harrington, Head
Department of Aeronautical
    Engineering
University of Cincinnati
Cincinnati 21, Ohio   (1)

Professor M. Hetenyi
The Technological Institute
Northwestern University
Evanston, Illinois   (1)

Professor P.G. Hodge
Department of Mechanics
Illinois Institute of Technology
Technology Center
Chicago 16, Illinois   (1)

Professor N.J. Hoff
Stanford University
Stanford, California   (1)

INVESTIGATORS ACTIVELY ENGAGED
IN RELATED RESEARCH (Continued)

Professor W. H. Koppmann, II
Department of Mechanical Engr.
John Hopkins University
Baltimore, Maryland   (1)

Professor Bruce G. Johnson
University of Michigan
Ann Arbor, Michigan   (1)

Professor J. Kempner
Department of Aeronautical Engr.
and Applied Mechanics
Polytechnic Institute of Brooklyn
99 Livingston Street
Brooklyn 2, New York   (1)

Professor H. L. Langhaar
Department of Theoretical
and Applied Mechanics
University of Illinois
Urbana, Illinois   (1)

Professor B. J. Lazan, Director
Engineering Experiment Station
University of Minnesota
Minneapolis 14, Minnesota   (1)

Professor E. H. Lee
Division of Applied Mathematics
Brown University
Providence 12, Rhode Island   (1)

Professor George H. Lee
Director of Research
Rensselaer Polytechnic Institute
Troy, New York

Mr. M. M. Lemcoe
Southwest Research Institute
8500 Culebra Road
San Antonio 6, Texas   (1)

Professor Paul Lieber
Geology Department
Rensselaer Polytechnic Institute
Troy, New York   (1)

Professor Hsu Lo
School of Engineering
Purdue University
Lafayette, Indiana   (1)

Professor R. D. Mindlin
Department of Civil Engineering
Columbia University
New York 27, N. Y.   (1)

Dr. A. Nadai
136 Cherry Valley Road
Pittsburgh 21, Pennsylvania   (1)

Professor William A. Nash
Department of Engineering Mechanics
University of Florida
Gainesville, Florida   (1)

Professor N. M. Newmark
Department of Civil Engineering
University of Illinois
Urbana, Illinois   (1)

Professor Aris Phillisps
Department of Civil Engineering
15 Prospect Street
Yale University
New Haven, Connecticut   (1)

Professor W. Prager, Chairman
Physical Sciences Council
Brown University
Providence 12, Rhode Island   (1)

Professor E. Reissner
Department of Mathematics
Massachusetts Institute of Technology
Cambridge 39, Massachusetts   (1)

Professor M. A. Sadowsky
Department of Mechanics
Rensselaer Polytechnic Institute
Troy, New York   (1)

Dr. B. W. Shaffer
Department of Mechanical Engineering
New York University
45 Fourth Avenue
New York 53, New York   (1)

Professor C. B. Smith
College of Arts and Science
Department of Mathematics
Walker Hall
University of Florida
Gainesville, Florida   (1)

Professor J. Stallmeyer
Department of Civil Engineering
University of Illinois
Urbana, Illinois   (1)

Professor Eli Sternberg
Brown University
Providence 12, Rhode Island   (1)

INVESTIGATORS ACTIVELY ENGAGED
IN RELATED RESEARCH (Continued)

Professor S.P. Timoshenko
School of Engineering
Stanford University
Stanford, California   (1)

Professor A.A. Velestos
Department of Civil Engineering
University of Illinois
Urbana, Illinois   (1)

Professor Enrico Volterra
Department of Engineering Mech.
University of Texas
Austin 12, Texas   (1)

Professor Dana Young
Yale University
New Haven, Connecticut   (1)

Project Staff   (10)